Università degli studi di Milano
Université de Bordeaux I

Dualities in étale cohomology

Alberto Merici

supervised by
Baptiste Morin

Academic year 2016/2017

Introduction

Amor ch'al cor gentil ratto s'apprende,
prese costui de la bella persona
che mi fu tolta; e 'l modo ancor m'offende

Amor ch'a nullo amato amar perdona,
mi prese del costui piacer sì forte,
che come vedi ancor non m'abbandona.

Dante Alighieri, *Inferno*, Canto V, vv. 100-105

Since its definition by Grothendieck and Deligne, étale cohomology has been seen as a "bridge" between arithmetics and geometry, i.e. something that generalizes purely geometric aspects, as sheaf cohomology, and purely arithmetic ones, as Galois cohomology.
One of the most important tools in order to calculate these invariants are duality theorems: roughly speaking, if Λ is a ring, for a cohomology theory $H^\bullet$ with values in Λ-mod and a corresponding compact supported cohomology theory $H_c^\bullet$, a duality is a collection of perfect pairings

$$H^r \times H_c^{n-r} \to H_c^n$$

where H_c^n is canonically isomorphic to a "nice" Λ-mod A. The first example of duality theorem one meets throughout the study of algebraic topology is probably Poincaré duality for De Rham cohomology (see [AT11]):

Theorem. *If X is an oriented differential manifold of dimension n, then the wedge product induces a perfect pairing of $\mathbb{R}$-vector spaces*

$$H^r(X, dR) \;\times\; H_c^{n-r}(X, dR) \longrightarrow H_c^n(X, dR) \cong \mathbb{R} \qquad (\eta,\ \psi) \longrightarrow \textstyle\int_X \eta \wedge \psi$$

On the other hand, in Galois cohomology one has the whole machinery of Tate dualities for finite, local and global fields: these are very important tools and I will recall them in the first chapter, mostly following [Mil06].

The aim of this mémoire is to generalize these theorems in the context of étale cohomology: the second chapter is dedicated to the proof of the Proper Base Change theorem, which I will prove following [Del], and from that proof I will obtain a nice definition of a cohomology with compact support. Then I will deduce Poincaré duality on smooth curves

over algebraically closed fields, as it is done in [Del] and [Fu11].
This will follow from an appropriate definition of the cup product pairing coming from the machinery of derived categories.
This approach leads almost immediately to a generalization of Poincaré duality on smooth curves over a finite field k, as it is done in [Mil16].
This is the link to arithmetics: in fact this theorem generalizes in some way to Artin-Verdier duality for global fields.
The aim of the second part is then to prove Artin-Verdier duality for global fields as it is done in [Mil06], although here, in the case of a number field with at least one real embedding, we need to refine the definition of cohomology with compact support in ordet to include the real primes too. This can be done in different ways, and I will briefly explain the approach given by [Mil16]. In the appendix, I will recall some results that are needed. I will recall results on the cohomology of the Idèle group, mostly following [CF67] and [Neu13], then results on the cohomology of topoi and the definition of étale cohomology with some important theorems involved. Most of the theorems are proved in [Tam12] or [Sta]. Then I will recall the definition of derived categories and some results needed in the mémoire, following [Har].
Finally, I will give one powerful application of Poincaré duality for algebraically closed fields: Grothendieck-Verdier-Lefschetz Trace formula and the consequent rationality of L-functions on curves over finite fields. I will give an idea of what it is done in [Del].

Acknowledgements

First of all, I would like to thank my family for always being there for me and for all the support they have shown throughout all my life: I would not be here without you. I thank mum and dad, for being the best parents I could ever imagine, my brother, for being the exact opposite of me, my grandmas and all my close and far relatives. A special thanks to my little cousin Sara, for being the strongest person I have ever known and I will probably ever know.

Secondly, I would like to thank my advisor Baptiste Morin for the incredible amount of time and effort he spent for my career and for listening to all my exposés, introducing me to this crazy and beautiful part of mathematics. He will always be of great inspiration to me.

I will also thank the Algant consortium for this great opportunity and all the professors I had the pleasure to meet, both in Milan and Bordeaux, and especially prof. Fabrizio Andreatta, I owe to him my passion for arithmetic.

Mathematics is for weirdos, so I thank all the weirdos I had the pleasure to meet in these years:
From Milan I thank Davide, Ale, Mosca, Edo, Fabio and Chiara for all the briscoloni, the Algant team "Senza nome": Isabella, Riccardo, Luca and Massimo for being the best travel companion in this adventure, for "Cuori Rubati", for staying late at university drawing commutative diagrams and for all the Skype sessions, Fra, for reminding me that sometimes analysis is nice, and Ema, for reminding that most of the times it's not, and last but not least, Lori, for being the beautiful person he always is.
From Bordeaux, I thank Danilo, for always being there when I needed a friend, Margarita, Corentin, Félicien, Alexandre, Nero and Thibault, for this incredible year together in the IMB.

A whole year alone in Bordeaux doing math can drive you insane, but luckily I met the right people to stay real: I thank every beautiful person I met this year, it would be impossible to mention everyone but I am sincerely happy that I have met all of you. I have to give a special thanks to MJ, for the beautiful trips we took together.

I would like to thank my longtime friends, who are still trying to figure out what is my position in the world: I thank the "Omega": Fede and his motto "There's more than one way to be an educated man", Ricca for being the craziest and most generous friend I will always have, and Giacomo for being the lovely jerk he will always be.
I also thank Corrado and Stefano, for being the high school friendship that is here for staying, even if I am far away.

Finally, a big thank goes to my roomies from Milan, for being the family every student far from home would dream of.

And last but not least, I would like to thank my best friend Edoardo for the late night phone calls, all the stupid messages and the support he has given me throughout all my life.

Contents

Chapter 1

Preliminaries: Tate duality

1.1 Local Tate duality

1.1.1 Tate cohomology groups

Definition 1.1.1. Let G be a finite group, C a G-module. We can define Tate cohomology groups as

$$\widehat{H}^r(G,M) = \begin{cases} H^r(G,M) & \text{if } r > 0 \\ M^G/N_G M \text{ where } N_G = \sum_{\sigma\in G}\sigma & \text{if } r = 0 \\ Ker(N_G)/I_G \text{ where } I_G = \{\sum_{\sigma\in G} a_\sigma \sigma \text{ with } \{\sum_{\sigma\in G} a_\sigma = 0\} & \text{if } r = -1 \\ H_{-r-1}(G,M) & \text{if } r < -1 \end{cases}$$

They can be computed using a *complete resolution*, i.e. an exact complex of finitely generated $\mathbb{Z}[G]$-modules

$$L_\bullet := \cdots L_1 \to L_0 \xrightarrow{d_0} L_{-1} \cdots$$

together with an element $e \in L_{-1}^G$ that generates the image of d_0, i.e. d_0 factors as

$$\begin{tikzcd} L_0 \arrow[rr, "d_0"] \arrow[dr, "\epsilon"] & & L_{-1} \\ & e\mathbb{Z} \arrow[ur] & \end{tikzcd}$$

such that $\epsilon(x)e = d_0(x)$. In particular, if we take a standard resolution $P_\bullet \xrightarrow{\alpha} \mathbb{Z}$ by free G-modules, we can consider $0 \to \mathbb{Z} \xrightarrow{\alpha_*} P_\bullet^*$ where

$$P_r^* = \mathrm{Hom}_{\mathbb{Z}}(P_r, \mathbb{Z})$$

So we have a complete resolution

$$\begin{tikzcd} P_{-1} \arrow[r] & P_0 \arrow[rr] \arrow[dr, "\alpha"] & & P_*^0 \arrow[r] & P_*^1 \\ & & \mathbb{Z} \arrow[ur, "\alpha_*"] & & \end{tikzcd}$$

And we can consider the cohomology:

$$\widehat{H}^r(G,M) = H^r(\mathrm{Hom}_G(L_\bullet, M))$$

Proposition 1.1.2. *For any G-equivariant pairing*

$$\alpha : M \oplus N \to Q$$

We have a unique cup product

$$(x,y) \mapsto x \cup y : \widehat{H}^r(G,M) \times \widehat{H}^s(G,N) \to \widehat{H}^{r+s}(G,Q)$$

such that

1. $dx \cup y = d(x \cup y)$
2. $x \cup dy = (-1)^{deg(x)} d(x \cup y)$
3. $x \cup y = (-1)^{deg(x)deg(y)}(y \cup x)$
4. $Res(x \cup y) = Res(x) \cup Res(y)$
5. $Inf(x \cup y) = Inf(x) \cup Inf(y)$

Proof. The idea is to generalize the construction for group cohomology, i.e. to construct a map

$$\Phi_{ij} : P_{i+j} \to P_i \otimes_{\mathbb{Z}} P_j$$

which composed with the map

$$\mathrm{Hom}(P_i, M) \otimes_{\mathbb{Z}} \mathrm{Hom}(P_j, N) \to \mathrm{Hom}(P_i \otimes_{\mathbb{Z}} P_j, M \otimes_{\mathbb{Z}} N)$$

$$\phi \otimes \psi \mapsto (a \otimes b \mapsto \phi(a) \otimes \psi(b))$$

gives a linear map

$$\mathrm{Hom}(P_i, M) \otimes_{\mathbb{Z}} \mathrm{Hom}(P_j, N) \to \mathrm{Hom}(P_{i+j}, Q)$$

such that

$$d(f \cup g) = df \cup g + (-1)^{deg(f)} f \cup dg$$

so the cup product extends uniquely on the cohomology classes. Hence the point is to show that such Φ_{ij} exists, that the induced cup-product respects the properties 1. – 5. and that such Φ is unique [CE16, Chap. XII, 4-5]

Theorem 1.1.3 (Tate-Nakayama). *Let G be a finite group and C be a G-module, $u \in H^2(G,C)$ such that:*

(a) $H^1(H,C) = 0$

(b) $H^2(H,C)$ *has order equal to that of H and is generated by $Res(u)$.*

Then, for any G-module M such that $Tor_1^{\mathbb{Z}}(M, C) = 0$, the induced cup-product

$$\widehat{H}^r(G, M) \times \widehat{H}^s(G, N) \to \widehat{H}^{r+s}(G, P)$$

defines an isomorphism

$$x \mapsto x \cup u : \widehat{H}^r(G, M) \to \widehat{H}^{r+2}(G, M \otimes C)$$

for all integers r.

Proof. [Ser62, IX]

1.1.2 Duality relative to class formation

Let G be a profinite group, C a G-module. A collection of isomorphisms

$$\{\mathrm{inv}_U : H^2(U, C) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}\}_{U \leq G \text{ open}}$$

is said to be a *class formation* if

(a) $H^1(U, C) = 0$ for all U

(b) For all pairs of subgroups $V \leq U \leq G$ with $[U : V] = n$ the following diagram commutes:

$$\begin{tikzcd}
H^2(U, C) \arrow[r] \arrow[d, "\mathrm{inv}_U"] & H^2(V, C) \arrow[d, "\mathrm{inv}_V"] \\
\mathbb{Q}/\mathbb{Z} \arrow[r, "n"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Remark 1.1.4. If V is normal in U, then the two conditions imply that we have an isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^2(U/V, C^V) \arrow[r] \arrow[d] & H^2(U, C) \arrow[r] \arrow[d, "\mathrm{inv}_U"] & H^2(V, C) \arrow[r] \arrow[d, "\mathrm{inv}_V"] & 0 \\
0 \arrow[r] & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "n"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The first exact sequence coming from Hochschield-Serre: we have from the seven terms exact sequence

$$H^1(V, C)^U \to H^2(U/V, C^V) \to Ker(H^2(U, C) \to H^2(V, C)) \to H^1(U/V, H^1(V, C))$$

and by hypothesis (a) we have

$$H^1(V, C)^U = 0 \qquad H^1(U/V, H^1(V, C)) = H^1(U/V, 0) = 0$$

so we have an isomorphism $H^2(U/V, C^V) \cong Ker(H^2(U, C) \to H^2(V, C))$. We call $u_{U/V}$ the element in $H^2(U/V, C^V)$ corresponding to $\frac{1}{n}$.

Lemma 1.1.5. *Let M be a G-module such that $Tor_1^{\mathbb{Z}}(M, C) = 0$. Then the map*

$$a \mapsto a \cup u_{G/U} : \widehat{H}^r(G/U, M) \to \widehat{H}^{r+2}(G/U, M \otimes_{\mathbb{Z}} C^U)$$

is an isomorphism for all open normal subgroups U of G and integers r.

Proof. Apply Tate-Nakayama to G/U, C^U and $u_{G/U}$

Theorem 1.1.6. *Let (G, C) a class formation, then there is a canonical map (the reci-procity map)*

$$rec_G : C^G \to G^{ab}$$

whose image in G^{ab} is dense and whose kernel is

$$\bigcap_U N_{G/U} C^U$$

Proof. Since $\widehat{H}^{-2}(G/U, \mathbb{Z}) = H_1(G/U, \mathbb{Z}) = (G/U)^{ab}$ and $\widehat{H}^0(G/U, C^U) = C^G/N_{G/U}C^U$, lemma 1.1.5 gives an isomorphism

$$(G/U)^{ab} \xrightarrow{\sim} C^G/N_{G/U}C^U$$

So taking the projective limit on the inverses of this maps we get a mono with dense image[1]

$$C^G/\bigcap_U N_{G/U}C^U \to G^{ab}$$

Hence rec_G is the corresponding map on C^G.

From now on, let G be a profinite group whose order is divisible by all the integers[2], (G, C) a class formation, M a finitely generated G-module and $\alpha^r(G, M) : \mathrm{Ext}^r_G(M, C) \to H^{2-r}(G, M)^*$ the maps induced by the pairings

$$\mathrm{Ext}^r_G(M, C) \times H^{2-r}(G, M) \to H^2(G, C) \cong \mathbb{Q}/\mathbb{Z}$$

In the particular case $M = \mathbb{Z}$:

a $\mathrm{Hom}_G(\mathbb{Z}, C) = C^G$ and $\mathrm{Hom}_G(H^2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_G(\mathrm{Hom}_G(G, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = (G^{ab})^{**} = G^{ab}$ for Pontrjagin duality, hence the map $\alpha^0(G, \mathbb{Z}) : C^G \to G^{ab}$ is rec_G (cfr [Ser62, XI, 3, Proposition 2])

b $\alpha^1(G, \mathbb{Z}) = 0$ since $H^1(G, \mathbb{Z}) = 0$

c $\mathrm{Hom}(\mathbb{Z}^G, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ and $\alpha^2(G, \mathbb{Z}) : H^2(G, M) \to \mathbb{Q}/\mathbb{Z}$ is inv_G

[1] In fact, if $C^G/\bigcap_U N_{G/U}C^U$ is compact Hausdorff then it is also epi, see for example [RZ00, 1.1.6 and 1.1.7]

[2] i.e. for all n there is an open subgroup U such that $[G : U] = n$. This of course makes every open subgroup divisible by all the integers

In the particular case $M = \mathbb{Z}/m\mathbb{Z}$, using the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

we have the long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(\mathbb{Z}/m\mathbb{Z}, C) \arrow[r] & C^G \arrow[r, "m"] & C^G \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{Ext}^1_G(\mathbb{Z}/m\mathbb{Z}, C) \arrow[r] & H^1(G,C) = 0 \arrow[r] & H^1(G,C) = 0 \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{Ext}^2_G(\mathbb{Z}/m\mathbb{Z}, C) \arrow[r] & H^2(G,C) \arrow[r] & H^2(G,C)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "m"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/m\mathbb{Z} \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^1(G,\mathbb{Z}) = 0 \arrow[r] & H^1(G,\mathbb{Z}) = 0 \arrow[r] & H^1(G,\mathbb{Z}/m\mathbb{Z}) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^2(G,\mathbb{Z}) = \mathrm{Hom}_G(G,\mathbb{Q}/\mathbb{Z}) \arrow[r, "m"] & H^2(G,\mathbb{Z}) = \mathrm{Hom}_G(G,\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(G,\mathbb{Z}/m\mathbb{Z})
\end{tikzcd}$$

So by dualizing the second one we get

$$0 \to H^0(G,\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\frac{1}{m}} \mathbb{Q}/\mathbb{Z} \to 0$$

$$H^2(G,\mathbb{Z}/m\mathbb{Z})^* \to G^{ab} \xrightarrow{m} G^{ab} \to H^1(G,\mathbb{Z}/m\mathbb{Z})^* \to 0$$

a' $\mathrm{Hom}_G(\mathbb{Z}/m\mathbb{Z}, C) = {}_m(C^G)$ and $H^2(G,\mathbb{Z}/m\mathbb{Z})^* \twoheadrightarrow {}_m(G^{ab})$, so the following diagram commutes

$$\begin{tikzcd}
{}_m(C^G) \arrow[r, "\alpha^0(G{,}\mathbb{Z}/m\mathbb{Z})"] \arrow[dr, "{}_m(rec_G)"'] & H^2(G,\mathbb{Z}/m\mathbb{Z})^\times \arrow[d, two heads] \\
& {}_m(G^{ab})
\end{tikzcd}$$

and if $H^3(G,\mathbb{Z}) = 0$ (e.g., if $cd(G) \leq 2$), then the vertical map is an isomorphism, hence in this case $\alpha^0(G,\mathbb{Z}/m\mathbb{Z}) = {}_m(rec_G)$

b' $\mathrm{Ext}^1_G(\mathbb{Z}/m\mathbb{Z}, C) = (C^G)_m$ and $H^1(G,\mathbb{Z}/m\mathbb{Z})^* = (G^{ab})^{(m)}$, so $\alpha^1(G,\mathbb{Z}/m\mathbb{Z}) = (rec_G)_m$

c' $\mathrm{Ext}^2_G(\mathbb{Z}/m\mathbb{Z}, C) = {}_mH^2(G,C)$ and $\mathbb{Z}/m\mathbb{Z}^* = \frac{1}{m}\mathbb{Z}/\mathbb{Z}$, so $\alpha^2(G,\mathbb{Z}/m\mathbb{Z}) = {}_m(inv_G)$

Lemma 1.1.7. *1. For $r \geq 4$, $Ext^r_G(M,C) = 0$*

2. For $r \geq 3$ and M torsion free, $Ext^r_G(M,C) = 0$

Proof. Recall that every finitely generated G-module can be solved as

$$0 \to M_1 \to M_2 \to M \to 0$$

with M_1 and M_2 finitely generated torsion free. Hence it's enough to prove to prove 2. Let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = \mathcal{H}om(M, \mathbb{Z})$, then $N \otimes_{\mathbb{Z}} C = \mathrm{Hom}_{\mathbb{Z}}(M, C)$ as G-modules, then we have the spectral sequence

$$H^p(G, \mathrm{Ext}^q_{\mathbb{Z}}(M, C)) \Rightarrow \mathrm{Ext}^{p+q}_G(M, C)$$

And since M is torsion free of finite type, let U be an open such that $M^U = M$, so M^U is a G/U-module torsion free of finite type, hence a torsion free of finite type $\mathbb{Z}$-module since G/U is finite, so it's a free $\mathbb{Z}$-module. So $\mathrm{Ext}^q_{\mathbb{Z}}(M, C) = \mathrm{Ext}^q_{\mathbb{Z}}(M^U, C) = 0$ for all $q > 0$, so the spectral sequence degenerates in degree 2 and we get

$$\mathrm{Ext}^p_G(M, C) = H^p(G, N \otimes_{\mathbb{Z}} C) = \varinjlim_{U \trianglelefteq G : N^U = N} H^p(G/U, N \otimes_{\mathbb{Z}} C^U)$$

So for Tate-Nakayama, $a \mapsto a \cup u_{G/H}$ gives for $r \geq 3$ the isomorphisms

$$H^{r-2}(G/U, N) \xrightarrow{\sim} H^r(G/U, N \otimes_{\mathbb{Z}} C^U)$$

Moreover, if $V \leq U$, we have by definition of u that $Inf(u_{G/U}) = [U : V]u_{G/V}$ and by definition of cup product we have $Inf(a \cup b) = Inf(a) \cup Inf(b)$, so we have a commutative diagram

$$\begin{tikzcd}
H^{r-2}(G/U, N) \arrow[r] \arrow[d, "{[U:V]Inf}"] & H^r(G/U, N \otimes_{\mathbb{Z}} C^U) \arrow[d, "Inf"] \\
H^{r-2}(G/V, N) \arrow[r] & H^r(G/V, N \otimes_{\mathbb{Z}} C^V)
\end{tikzcd}$$

But since $H^{r-2}(G/U, N)$ is torsion and the order of U is divisible by all the integers, we have that if $r - 2 \geq 1$

$$\varinjlim_{U \trianglelefteq G : N^U = N} H^{r-2}(G/U, N) = 0$$

Theorem 1.1.8. *(a) The map $\alpha^r(G, M)$ is bijective for all $r \geq 2$, and $\alpha^1(G, M)$ is bijective for all torsion-free M. In particular $Ext^r_G(M, C) = 0$ for $r \geq 3$.*

(b) The map $\alpha^1(G, M)$ is bijective for all M if $\alpha^1(U, \mathbb{Z}/m\mathbb{Z})$ is bijective for all open subgroups U of G and all m:

(c) The map $\alpha^0(G, M)$ is surjective (respectively bijective) for all finite M if in addition $\alpha^0(U, \mathbb{Z}/m\mathbb{Z})$ is surjective (respectively bijective) for all U and all m

Proof. For lemma 1.1.7, the theorem is true for $r \geq 4$- Suppose now that G acts trivially on M, so $M = \mathbb{Z}^I \bigoplus \oplus_i \mathbb{Z}/m_i\mathbb{Z}$, hence

$$\mathrm{Ext}^r_G(M, C) = (\oplus_I \mathrm{Ext}^r_G(\mathbb{Z}, C)) \bigoplus (\oplus_i \mathrm{Ext}^r_G(\mathbb{Z}/m_i\mathbb{Z}, C))$$

and

$$H^r(G,M) = (\oplus_I H^r(G,\mathbb{Z}))\bigoplus(\oplus_i H^r(G,\mathbb{Z}/m_i\mathbb{Z})$$

Hence the theorem is true for $r \leq 2$ and M with trivial action. Moreover, $\mathrm{Ext}^3_G(\mathbb{Z},C) = 0$ for lemma 1.1.7 since $\mathbb{Z}$ is torsion-free, so we have an exact sequence

$$\mathrm{Ext}^2_G(\mathbb{Z},C) \xrightarrow{m} \mathrm{Ext}^2_G(\mathbb{Z},C) \to \mathrm{Ext}^3_G(\mathbb{Z}/m\mathbb{Z},C) \to 0$$

But since $\mathrm{Ext}^2(\mathbb{Z},C) = H^2(G,C) \cong \mathbb{Q}/\mathbb{Z}$ is divisible, $\mathrm{Ext}^3(\mathbb{Z}/m\mathbb{Z},C) = 0$. So the theorem is true if the action on M is trivial.
Consider now a general M. Consider U a normal open subgroup of G such that $M^U = M$ (it exists since M is finitely generated), and take $M_* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/U],M) = \mathbb{Z}[G/U]\otimes_{\mathbb{Z}} M$. Then the spectral sequence

$$\mathrm{Ext}^p_{G/U}(\mathbb{Z}[G/U],\mathrm{Ext}^q_U(M,C)) \Rightarrow \mathrm{Ext}^{p+q}_G(M_*,C)$$

degenerates in degree 2 so $\mathrm{Ext}^r_U(M,C) = \mathrm{Hom}_{\mathbb{Z}[G/U]}(\mathbb{Z}[G/U],\mathrm{Ext}^r_U(M,C)) = \mathrm{Ext}^r_G(M_*,C)$. On the other hand

$$\mathrm{Ext}^p_{G/U}(\mathbb{Z}[G/U],H^q(U,M)) \Rightarrow \mathrm{Ext}^{p+q}_G(\mathbb{Z}[G/U],M)$$

degenerates in degree 2 so $H^r(U,M) = \mathrm{Hom}_{\mathbb{Z}[G/U]}(\mathbb{Z}[G/U],H^r(U,M)) = \mathrm{Ext}^r_G(\mathbb{Z}[G/U],M)$ and for lemma C.8.4

$$\mathrm{Hom}_G(\mathbb{Z}[G/U],M) = \mathrm{Hom}_G(\mathbb{Z},\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/U],M))$$

Since $\mathbb{Z}[G/U]$ is projective and finitely generated as $\mathbb{Z}$-module, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/U],\_) = \mathcal{H}om_{\mathbb{Z}}(\mathbb{Z}[G/U],\_)$ is exact and sends injectives to $\mathrm{Hom}_G(\mathbb{Z},\_)$-acyclics, hence

$$H^r(U,M) = \mathrm{Ext}^r_G(\mathbb{Z},M_*) = H^r(G,M_*)$$

So we have the exact sequence

$$0 \to M \to M_* \to M_1 \to 0$$

which induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^r_G(M_1,C) \arrow[r] \arrow[d, "\alpha^r(G,M_1)"] & \mathrm{Ext}^r_U(M,C) \arrow[r] \arrow[d, "\alpha^r(U,M)"] & \mathrm{Ext}^r_G(M,C) \arrow[r] \arrow[d, "\alpha^r(G,M)"] & \mathrm{Ext}^{r+1}_G(M_1,C) \arrow[d, "\alpha^{r+1}(G,M_1)"] \\
H^{2-r}(G,M_1)^\times \arrow[r] & H^{2-r}(U,M)^* \arrow[r] & H^{2-r}(G,M)^* \arrow[r] & H^{1-r}(G,M_1)^\times
\end{tikzcd}$$

Since $\alpha^3(U,M)$, $\alpha^4(G,M_1)$ and $\alpha^4(U,M)$ are isomorphisms, by five lemma $\alpha^3(G,M)$ is surjective, and since it holds for all M, also $\alpha^3(G,M_1)$ is surjecitve, hence five lemma shows that $\alpha^3(G,M)$ is an isomorphism. The same argument shows that $\alpha^2(G,M)$ is an isomorphism. If M is torsion free, then M_* and M_1 are also torsion free and $\alpha^1(U,M)$ is an isomorphism, hence by the same argument $\alpha^1(G,M)$ is an isomorphism, so (a) is proved, and by the same idea we get in general (b) and (c) □

1.1.3 Dualities in Galois cohomology

$G = \widehat{\mathbb{Z}}$

Let G be isomorphic to $\widehat{\mathbb{Z}}$ and $C = \mathbb{Z}$. Then it is generated by an element σ and all the open subgroups of G are generated by σ^m. We have an isomorphism $H^2(U,\mathbb{Z}) \cong H^1(U,\mathbb{Q}/\mathbb{Z})$ induced by the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

So we define the reciprocity map to be the composite of this isomorphism with

$$H^1(U,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_U(U,\mathbb{Q}/\mathbb{Z}) \xrightarrow{f \mapsto f(\sigma^m)} \mathbb{Q}/\mathbb{Z}$$

This is clearly a class formation and depends on the choice of σ. The reciprocity map is injective but not surjective: it is the inclusion $n \mapsto \sigma^n$. Since for all $U \leq \widehat{\mathbb{Z}}$ we have ${}_m(\mathbb{Z}^U) = {}_m(\mathbb{Z}) = 0$ and $H^2(U,\mathbb{Z}/m\mathbb{Z}) = 0$ because $cd(\widehat{\mathbb{Z}}) = 1$, so $\alpha^0(U,\mathbb{Z}/m\mathbb{Z})$ is an isomorphism for all U and all m. Moreover, $(\mathbb{Z}^U)_m = \mathbb{Z}/m\mathbb{Z}$ and $(\widehat{\mathbb{Z}}^{ab})_m = \widehat{\mathbb{Z}}/m\widehat{\mathbb{Z}}$, so $\alpha^1(U,\mathbb{Z}/m\mathbb{Z})$ is an isomorphism for all U and all m. Hence $\alpha^r(G,M)$ is an isomorphism for all finitely generated M and for all $r \geq 1$ and $\alpha^0(G,M)$ is an isomorphism for all finite M.
When M is finite, $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) = 0$ and for the exact sequence we get $\mathrm{Ext}^r_{\mathbb{Z}}(M,\mathbb{Z}) = 0$ for all $r \neq 1$ and

$$\mathrm{Ext}^1_{\mathbb{Z}}(M,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) =: M^*$$

Hence, using the spectral sequence

$$H^p(G,\mathrm{Ext}^q_{\mathbb{Z}}(M,\mathbb{Z})) \Rightarrow \mathrm{Ext}^{p+q}_G(M,\mathbb{Z})$$

we get $\mathrm{Ext}^r_G(M,\mathbb{Z}) = H^{r-1}(G,M^*)$, so we have a perfect pairing

$$H^r(G,M) \times H^{1-r}(G,M^*) \to \mathbb{Q}/\mathbb{Z}$$

Moreover, if M is finitely generated, then $\mathrm{Hom}_G(M,\mathbb{Z})$ is finitely generated, and if we consider U such that $M^U = M$, then Hochschield-Serre gives us

$$0 \to H^1(G/U,M) \to H^1(G,M) \to H^1(U,M)$$

Then $H^1(G/U,M)$ is finite since G/U is finite, and since $M^U = M$

$$H^1(U,M) = Hom_{cts}(U,M) = Hom_{cts}(\widehat{\mathbb{Z}},\mathbb{Z}^I \oplus \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$$

is also finite, then $H^1(G,M)$ is finite, so $H^1(G,M)^*$ is finite and via α^1 we get $\mathrm{Ext}^1(\mathbb{Z},M)$ is finite.
In particular, we can summarize

Theorem 1.1.9. *Let $G \cong \widehat{\mathbb{Z}}$, M a finite G-module, $M^* = Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ the Pontryagin dual, we have a perfect pairing of finite groups*

$$H^r(G,M) \times H^{1-r}(G,M^*) \to \mathbb{Q}/\mathbb{Z}$$

If now M is finitely generated, if we apply $\_\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ to the exact sequence of theorem 1.1.8 we get

$$0 \to \mathrm{Hom}_G(M_1,\mathbb{Z})^\wedge \to \mathrm{Hom}_U(M,\mathbb{Z})^\wedge \to \mathrm{Hom}_G(M,\mathbb{Z})^\wedge \to \mathrm{Ext}^1(M_1,\mathbb{Z})$$

We have that on the completion $\widehat{\alpha}^0(G,\mathbb{Z}) = \widehat{rec_G} : \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ is an isomorphism (in fact, it is the identity), so $\widehat{\alpha}^0(G,M)$ is an isomorphism if G acts trivially on M, and we can conclude by the same way of theorem 1.1.8 that $\widehat{\alpha}^0(U,M)$ is an isomorphism using the exact sequence on the completion, hence $\widehat{\alpha}^0(G,M)$ is an isomorphism for all M finitely generated. In particular, we have

Theorem 1.1.10. *Let $G \cong \widehat{\mathbb{Z}}$, M a finitely generated G-module, $M^* = Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ the Pontryagin dual, we have*

1. *$\alpha^0(G,M) : Hom_G(M,\mathbb{Z})^\wedge \xrightarrow{\sim} H^2(G,M)^*$*
2. *$Ext^1_G(M,\mathbb{Z}) \xrightarrow{\sim} H^1(G,M)^*$ are finite groups*
3. *$Ext^2_G(M,\mathbb{Z}) \xrightarrow{\sim} Hom_{\mathbb{Z}}(M^G,\mathbb{Q}/\mathbb{Z})$*
4. *$Ext^r_G(M,\mathbb{Z}) = 0$ for $r \geq 3$*

Local Fields

Let K be a local field, $\overline{K}$ a fixed separable closure and K_0 the maximal unramified extension. $G = Gal(\overline{K}/K)$ the absolute Galois group, $I = Gal(\overline{K}/K_0)$ the inertia subgroup and $C = \overline{K}^\times$. Then Hochschield-Serre induces the exact sequence

$$H^1(I,\overline{K}^\times) = 0 \to H^2(G/I,K_0^\times) \to Ker(H^2(G,\overline{K}^\times) \to H^0(G/I,H^2(I,\overline{K}^\times)) \to H^1(G/I,H^1(I,\overline{K}^\times)) = 0$$

and $H^2(I,\overline{K}^\times) = 0$ for the local class field theory ([Ser62, X, 7, Proposition 11]), so the inflation map is an isomorphism

$$H^2(G/I,K_0^\times) \xrightarrow{\sim} H^2(G,\overline{K}^\times)$$

Since for every finite unramified extension L/K, if $U_L = \mathcal{O}_L^\times$, then $H^r(Gal(L/K),U_L) = 0$, so the exact sequence

$$0 \to U_L \to L^\times \to \mathbb{Z} \to 0$$

gives an isomorphism passing to the limit

$$H^2(G/I,K_0^\times) \cong H^2(\widehat{\mathbb{Z}},\mathbb{Z}) \xrightarrow{Inv_{G/I}} \mathbb{Q}/\mathbb{Z}$$

Where $Inv_{G/I}$ is given by the previous example with $\sigma = Frob$. Then this gives rise to a class formation ([Mil97, III, Proposition 1.8]) with reciprocity map $rec_G : K^\times \to G^{ab}$ injective with dense image, the norm groups are the open subgroups of G by local class field theory ([Mil97]).
Consider U an open subgroup of G, $F = K^U$ the corresponding finite abelian extension of

K. By local class field theory, $\widehat{F^\times} \xrightarrow{\sim} U^{ab}$ is the completion morphism, hence we have a morphism of left exact sequences

$$\begin{tikzcd} 0 \arrow[r] & {}_mF^\times \arrow[r] \arrow[d, "\alpha^0(U,\mathbb{Z}/m\mathbb{Z})"] & F^\times \arrow[r, "m"] \arrow[d] & F^\times \arrow[d] \\ 0 \arrow[r] & {}_mU^{ab} \arrow[r] & U^{ab} \arrow[r, "m"] & U^{ab} \end{tikzcd}$$

Then $\alpha^0(U, \mathbb{Z}/m\mathbb{Z})$ is the completion morphism, hence it is injective with dense image, and since ${}_mF^\times = \mu_m(F)$ is finite, it is an isomorphism. Moreover, consider the cokernel

$$\begin{tikzcd} F^\times \arrow[r, "m"] \arrow[d] & F^\times \arrow[r] \arrow[d] & F^\times_m \arrow[r] \arrow[d, "\alpha^1(U,\mathbb{Z}/m\mathbb{Z})"] & 0 \\ U^{ab} \arrow[r, "m"] & U^{ab} \arrow[r] & U^{ab}_m \arrow[r] & 0 \end{tikzcd}$$

We have the following morphism of exact sequences given by the completion

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_F^\times \arrow[r] \arrow[d] & F^\times \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_F^{ab} \arrow[r] & U^{ab} \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd}$$

The first one is an isomorphism since $\mathcal{O}_F \cong k \times \mathfrak{M}$ is a topological isomorphism, hence $\mathcal{O}_F^\times$ is complete. Hence, since $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$ is injective, we have an induced exact sequence on the cokernels

$$\begin{tikzcd} 0 \arrow[r] & (\mathcal{O}_F^\times)_m \arrow[r] \arrow[d] & (F^\times)^{(m)} \arrow[r] \arrow[d, "\alpha^1(U,\mathbb{Z}/m\mathbb{Z})"] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (I_F^{ab})_m \arrow[r] & (U^{ab})_m \arrow[r] & \widehat{\mathbb{Z}}/m\widehat{\mathbb{Z}} = \mathbb{Z}/m\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

And since the two external maps are isomorphisms, we have that $\alpha^1(U, \mathbb{Z}/m\mathbb{Z})$ are isomorphisms for all U and all m hence for theorem 1.1.8

$$\alpha^r(G, M) : \mathrm{Ext}^r_G(M, \overline{K}^\times) \to H^{2-r}(G, M)$$

is an isomorphism for all finitely generated G-modules M for all $r \geq 1$, and if M is finite $\alpha^0(G, M)$ also is. If now M has torsion part prime to $char(K)$, since $\overline{K}^\times$ is divisible by all primes different from the characteristic of K, we have $\mathrm{Ext}^r_{\mathbb{Z}}(M, \overline{K}^\times) = 0$ for $r \geq 1$, so we have by the degenerating Ext sequence

$$H^r(G, \mathrm{Hom}_{\mathbb{Z}}(M, \overline{K}^\times)) \cong \mathrm{Ext}^r_G(M, \overline{K}^\times)$$

So we can summarize

Theorem 1.1.11 (Local Tate Duality). *If K is a local field, M a finite G_K-module, $M^D = Hom_{\mathbb{Z}}(M, \overline{K}^\times)$ the Cartier dual, then we have a perfect pairing*

$$H^r(G_K, M) \times H^{2-r}(G_K, M^D) \to \mathbb{Q}/\mathbb{Z}$$

Moreover, if M is finite with order prime to $char(K)$ then $Ext^r_G(M, \overline{K}^\times)$ and $H^r(G, M)$ are finite

Proof. The only assertion who needs a proof is the finiteness:
We know by Kummer exact sequence that

$$0 \longrightarrow H^0(G, \mu_n(\overline{K}^\times)) = \mu_n(K^\times) \longrightarrow K^\times \xrightarrow{(\_)^n} K^\times \longrightarrow H^1(G, \mu_n(\overline{K}^\times)) \longrightarrow H^1(G, \overline{K}^\times) = 0 \longrightarrow H^1(G, \overline{K}^\times) = 0 \longrightarrow H^2(G, \mu_n(\overline{K}^\times) \longrightarrow H^2(G, \overline{K}^\times) = \mathbb{Q}/\mathbb{Z} \longrightarrow H^2(G, \overline{K}^\times) = \mathbb{Q}/\mathbb{Z}$$

Hence we get

- $H^1(G, \mu_n(\overline{K}^\times)) = K^\times/K^{\times n}$
- $H^2(G, \mu_n(\overline{K}^\times)) = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$
- $H^r(G, \mu_n(\overline{K}^\times)) = 0$ for $r > 2$ $(cd(G) \leq 2)$

In particular, they are all finite.
Consider now a finite extension L/K which contains all the m^{th} toots of 1, with m dividing the order of M and such that $Gal(\overline{K}/L)$ acts trivially on M, so M as a $Gal(\overline{K}/L)$-module is isomorphic to a finite sum of copies of μ_m, so $H^r(Gal(\overline{K}/L), M)$ is finite. We can use Hochschield-Serre's seven-terms exact sequence, if $N = Ker(H^2(G, M) \to H^2(Gal(\overline{K}/L), M)^{Gal(L/K)}$):

$$0 \longrightarrow H^1(Gal(L/K), M) \longrightarrow H^1(G, M) \longrightarrow H^1(Gal(\overline{K}/L), M)^{Gal(L/K)} \longrightarrow H^2(Gal(L/K), M) \longrightarrow N \longrightarrow H^1(Gal(L/K), H^1(Gal(\overline{K}/L), M))$$

$$0 \longrightarrow N \longrightarrow H^2(G, M) \longrightarrow H^2(Gal(\overline{K}/L), M)^{Gal(L/K)} \longrightarrow 0$$

So $H^1(G, M)$ and $H^1(G, M)$ are finite since $Gal(L/K)$ is a finite group, and by duality $\mathrm{Ext}^r_G(M, \overline{K}^\times)$ is finite. □

We can enounce local Tate duality in its general form:

Theorem 1.1.12. *Let M be a finitely generated G-module whose torsion subgroup has order prime to $char(K)$. Then for $r \geq 1$ we have isomorphisms*

$$H^r(G, M^D) \to H^{2-r}(G, M)^*$$

and an isomorphism of profinite groups

$$H^0(G, M^D)^\wedge \to H^2(G, M)^*$$

Moreover, $H^1(G, M)$ and $H^1(G, M^D)$ are finite groups.

Proof. To prove the finiteness, by the previous result we can assume M torsion free. Let L/K be a finite extension such that $Gal(\overline{K}/L)$ acts trivially on M. Then the inflation-restirction exact sequence

$$0 \to H^1(Gal(L/K), M) \to H^1(G, M) \to H^1(Gal(\overline{K}/L), M)^{Gal(L/K)}$$

And since $H^1(Gal(\overline{K}/L), M) = \mathrm{Hom}_{cts}(Gal(\overline{K}/L), M) = 0$ since $M = \mathbb{Z}^m$ and G is compact. This shows that $H^1(G, M)$ is finite, and by duality $H^1(G, M^D)$ is finite.
Now, we have that

$$\widehat{\alpha}^0(G, \mathbb{Z}) = \widehat{rec}_G : \widehat{K}^\times \to G$$

is an isomorphism (rec_G is injective with dense image), hence $\widehat{\alpha}^0(G, M)$ is an isomorphism if G acts trivially on M, so we can conclude by the same way as theorem 1.1.10. □

Henselian fields

Let R be an Henselian DVR with finite residue field, and let K be its fraction field. The valuation lifts uniquely to $\overline{K}$ and so $Gal(\overline{K}/K) = Gal(\widehat{\overline{K}}/\widehat{K})$. So we have:

Proposition 1.1.13. *There is a canonical isomorphism $inv_K : Br(K) \cong H^2(Gal(K_0/K), K_0^\times) \cong \mathbb{Q}/\mathbb{Z}$ which respects the class formation axiom:*

Sketch of proof. First, we need to show that $Br(K_0^\times) = 0$, then we have the first isomorphism using the exact sequence

$$0 \to H^2(G(K_0/K), K_0^\times) \to Br(K) \to Br(K_0)$$

Using the split exact sequence of $Gal(K_0/K)$-modules

$$0 \to R_0^\times \to K_0^\times \to \mathbb{Z} \to 0$$

shows that $H^2(Gal(K_0/K), K_0^\times) \to H^2(Gal(K_0/K), \mathbb{Z})$ is surjective, and by some trick we can show that its kernel is zero. Since $H^2(Gal(K_0/K), \mathbb{Z}) \cong H^1(Gal(K_0/K), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ canonically, we have the isomorphism.
If now F/K is a finite separable extension, R_F is again an Henselian DVR with finite residue field, and by definition one has

$$inv_F(Res(a)) = [F : K] inv_K(a)$$

For the details, see [Mil06, Ch. I, Appendix A]

So again $(G_K, \overline{K}^\times)$ is a class formation, hence we have a reciprocity law

$$rec_G : K^\times \to G^{ab}$$

whose kernel is $\bigcap_{L/K\text{finite separable}} N_{L/K}L^\times$, hence we have an isomorphism

$$\varprojlim K^\times/N_{L/K} \xrightarrow{\sim} G^{ab}$$

If now we ask R to be also *excellent*, i.e. such that the completion $\widehat{K}/K$ is separable over K, and that its residue field is finite. (Notice that the Henselization of a local ring at a prime in a global field satisfies this hypothesis).

Lemma 1.1.14. *(i) Every finite separable extension of $\widehat{K}$ is of the form $\widehat{F}$ for a finite separable extension F/K. Moreover $[F : K] = [\widehat{F} : \widehat{K}]$*

(ii) K is algebraically closed in $\widehat{K}$.

Proof. (i) It follows from Krasner's Lemma: take $\widehat{F} = \widehat{K}[\alpha]$ and let f_α be its minimal polynomial, consider $f_n(T)$ a sequence in $K[T]$ converging to $f_\alpha(T)$ and let $F = K[\beta]$ for β a root of f_n for n big enough

(ii) Take $\alpha \in \widehat{K}$ integral over R. Take f its minimal polynomial over R, since $\widehat{R}$ is a DVR, hence integrally closed, f has a root in $\widehat{R}$, and again from Krasner's Lemma we conclude that f has a root in R, hence $\alpha \in R$.

Remark 1.1.15. From the separability and (ii), we conclude that $\widehat{K}$ is linearly disjoint from K^{alg} (see [Lan72, III, Thm 2])

Using this result, one can see that

$$NF^\times = N\widehat{F}^\times \cap K^\times$$

So we can conclude from the existence theorem of local class field theory that

Theorem 1.1.16 (Existence theorem for excellent Henselian DVR with finite residue field)**.** *The norm subgroups of $K^\times$ are exactly the open subgroups of $K^\times$ of finite index*

Hence $\alpha^0(G, \mathbb{Z})$ defines again an isomorphism

$$\widehat{\alpha}^0(G, \mathbb{Z}) : (K^\times)^\wedge \to G^{ab}$$

We get that $\alpha^0(G, \mathbb{Z}) = \widehat{rec}$ is again an isomorphism and $\alpha^0(G, \mathbb{Z}/m\mathbb{Z}) = 0$ is an isomorphism as in the previous examples. α^1 So we can now generalize local Tate duality:

Theorem 1.1.17. *Let K is the fraction field of an excellent Henselian DVR with finite residue field, M a finite G_K-module whose torsion subgroup is prime to $char(K)$, $M^D = Hom_{\mathbb{Z}}(M, \overline{K}^\times)$ the Cartier dual. Then for $r \geq 1$ we have isomorphisms*

$$H^r(G, M^D) \to H^{2-r}(G, M)^*$$

and an isomorphism

$$H^0(G, M^D)^\wedge \to H^2(G, M)^*$$

Moreover, $H^1(G, M)$ and $H^1(G, M^D)$ are finite groups.

Proof. We only need to show that for every m prime to $char(K)$, $\alpha^0(U, \mathbb{Z}/m\mathbb{Z})$ and $\alpha^1(U, \mathbb{Z}/m\mathbb{Z})$ are isomorphisms for all U and all m. So take F the finite extension of K corresponding to U. R_F is again an Henselian local ring, so we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & R_F^\times \arrow[r] \arrow[d] & F^\times \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widehat{R^\times}_F \arrow[r] & U^{ab} \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd}$$

Again, we have as in theorem 1.1.12 that

$$_m(R_F^\times) \to {}_m(\widehat{R_F^\times})$$

is injective with dense image, but $(R_F^\times)_m$ is finite, hence it is an isomorphism, and we conclude that $\alpha^0(U, \mathbb{Z}/m\mathbb{Z}) : {}_mF^\times \to {}_mU^{ab}$ is an isomorphism. Then, since $\widehat{R_F^\times}$ is divisible by all primes $\neq char(k)$ because it is Henselian, we have that $R_m^\times \to \widehat{R}_m^\times$ is an isomorphism, so $\alpha^1(U, \mathbb{Z}/m\mathbb{Z}) : F_m^\times \to U_m^{ab}$ is an isomorphism for all m prime to $char(k)$, and we conclude by the same way as theorem 1.1.12 □

In particular, we have that:

Theorem 1.1.18 (Generalized local Tate duality)**.** *If M is finite, we have a perfect pairing*

$$H^r(G_K, M) \times H^{2-r}(G_K, M^D) \to \mathbb{Q}/\mathbb{Z}$$

Moreover, $Ext^r(M, \overline{K}^\times)$ and $H^r(G, M)$ are finite

Archimedean fields

We have a duality theorem for $K = \mathbb{R}$:

Theorem 1.1.19. *Let $G = Gal(\mathbb{C}/\mathbb{R})$. For every finitely generated module M with dual $M^D = Hom(M, \mathbb{C}^\times)$, we have a nondegenerate pairing of finite groups:*

$$\widehat{H}^r(G, M^D) \times \widehat{H}^{2-r}(G, M) \to H^2(G, \mathbb{C}^\times) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

Proof. Let M be finite, then G acts only on the 2-primary component of M, so we can suppose M 2-primary, and using the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to M \to M' \to 0$$

using induction on the order of M we need to prove it for $M = \mathbb{Z}/2\mathbb{Z}$ with trivial action. Then $M^D = \mathbb{Z}/2\mathbb{Z}$ and since G is cyclic we have

0. $\widehat{H}^0(G, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}/N_{\mathbb{C}/\mathbb{R}}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$
1. $\widehat{H}^1(G, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

If $M = \mathbb{Z}$, then $\mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times$, so

1. $\widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/N_{\mathbb{C}/\mathbb{R}}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $\widehat{H}^0(G, \mathbb{C}^\times) = \mathbb{R}^\times/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$
2. $\widehat{H}^1(G, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, $\widehat{H}^1(G, \mathbb{C}^\times) = 0$ for Hilbert 90.

And if $M = \mathbb{Z}[G]$ every group is 0 ($\mathbb{Z}[G]$ is $\mathbb{Z}[G]$-projective).
Combining all this we have the result. □

1.2 Global Tate duality

Consider K a global field, S a non empty set of places containing all the nonarchimedean if k is a number field. If F/K is a finite extension, we will denote by S_F the set of places lying over S, and if the context is clear just by S.
K_S would be the maximal subextension of $\overline{K}$ which is ramified only over S, which is Galois, and let G_S be its Galois group.
Let now $\mathcal{O}_S$ be the ring of S-integers:

$$\mathcal{O}_S := \bigcap_{v \notin S} \mathcal{O}_v = \{a \in K : v(a) \geq 0 \text{ for all } v \notin S\}$$

For each place, choose an embedding $\overline{k} \to \overline{k}_v$ and an isomorphism of $G_v = Gal(\overline{k}_v/k_v)$ to the decomposition subgroup of G.
Consider P a set of prime numbers ℓ such that for all n ℓ^∞ divides the degree of K_S over K. If $K = k(X)$ is a function field, then since $\overline{k}K \subseteq K_S$, then $\mathbb{P}$ is the set of all primes. It is known ([CC09, Cor 5.2]) that if S contains all the places over at least two primes of $\mathbb{Q}$, then P is the set of all primes, but in general we have no idea how large P is.
Let F/K be a finite extension contained in K_S. Then define:

- $\mathbb{I}_F$ the IdÃĺle group of F, $\mathbb{I}_{F,S}$ the S-idÃĺle group $\prod_{v \in S}^{\mathcal{O}_v^\times} F_v^\times$, with the canonical inclusion $\mathbb{I}_{F,S} \hookrightarrow \mathbb{I}_F$ given by
$$(a_v) \in \mathbb{I}_{F,S} \Leftrightarrow a_v = 1 \text{ for all } v \notin S$$
- $\mathcal{O}_{F,S}$ the ring of S-integers of F, i.e. $\bigcap_{v \notin S} \mathcal{O}_v$ (the normal closure of $\mathcal{O}_{K,S}$ in F) and $E_{F,S} := \mathcal{O}_{F,S}^\times$ the S-units and $Cl_{F,S}$ its class group.
- $C_{F,S} := \mathbb{I}_{F,S}/E_{F,S}$ the S-idÃĺle class group
- $\mathbb{U}_{F,S} := \prod_{w \notin S} \mathcal{O}_w$ with the canonical inclusion $\mathbb{U}_{F,S} \hookrightarrow \mathbb{I}_F$
$$(a_v) \in \mathbb{U}_{F,S} \Leftrightarrow a_v = 1 \text{ for all } v \in S \text{ and } a_v \in \mathcal{O}_v^\times \text{ for all } v \notin S$$
- $C_S(F) = C_F/\mathbb{U}_{F,S}$

Taking the direct limit over F we can define $\mathbb{I}_S$, $\mathcal{O}_S$, E_S, C_S, $\mathbb{U}_S$.

Remark 1.2.1. If S contains all primes, then $K_S = \overline{K}$, $G_S = G_K$ and P contains all the primes. The object just defined are respectively $\mathbb{I}_F$, F, $F^\times$, C_F and 1.

We know by [CF67, VII] (see Chapter A) that if F is a global field, then (G_F, C_F) is a class formation. We want to generalize this to (G_S, C_S), so we need to generalize theorem 1.1.8.

1.2.1 P-class formation

Let P be a set of prime numbers, G a profinite group, C a G-module. Then (G,C) is a P-class formation if for all open subgroups U of G, $H^1(U,C) = 0$ and there is a family of monomorphisms

$$inv_U : H^2(U,C) \to \mathbb{Q}/\mathbb{Z}$$

such that:

1. For all pairs of subgroups $V \leq U \leq G$ with $[U:V] = n$ the following diagram commutes:

$$\begin{tikzcd} H^2(U,C) \arrow[r] \arrow[d, "\mathrm{inv}_U"] & H^2(V,C) \arrow[d, "\mathrm{inv}_V"] \\ \mathbb{Q}/\mathbb{Z} \arrow[r, "n"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

2. If V is normal in U, the map

$$inv_{U/V} : H^2(U/V.C^V) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is an isomorphism

3. For all $\ell \in P$, then the map on the ℓ-primary components $inv_U : H^2(U,C)(\ell) \to (\mathbb{Q}/\mathbb{Z})(\ell)$ is an isomorphism

Since, if M is finitely generated, $\mathrm{Ext}^r_G(M,N)$ is torsion for $r > 1$, we can apply the same method as for the class formation to the ℓ-primary components and get the theorem:

Theorem 1.2.2. *Let (G,C) be a P-class formation, $\ell \in P$ and M a finitely generated G-module.*

(a) The map $\alpha^r(G,M)(\ell) : Ext^r_G(M,N)(\ell) \to H^{2-r}(G,M)^*(\ell)$ is an isomorphism for $r \geq 2$ and if M is torsion free, also for $r = 1$.

(b) The map $\alpha^1(G,M)(\ell)$ is an isomorphism if $\alpha^1(U,\mathbb{Z}/\ell^n\mathbb{Z})(\ell)$ is an isomorphism for all U open subgroup and $n \in \mathbb{N}$.

(c) The map $\alpha^0(G,M)$ is epi (resp. an isomorphism) for all M finite ℓ-primary group if in addiction $\alpha^0(U,\mathbb{Z}/\ell^n\mathbb{Z})$ is epi (resp. an isomorphism) for all U open subgroup and $n \in \mathbb{N}$.

Consider $C_S(F) = C_F/\mathbb{U}_{F,S}$, we want to show that (G_S, C_S) is a P-class formation, and moreover that $C_S^{Gal(K_S/F)} = C_S(F)$. We have that if S contains all the primes then (G,C) is a class formation, so a P class formation, and $C_S(F) = C_F$.

Lemma 1.2.3. *There is an exact sequence*

$$0 \to C_{F,S} \to C_S(F) \to Cl_{F,S} \to 0$$

Proof. Notice that in $\mathbb{I}_F$ we have $\mathbb{U}_{F,S} \cap F^\times = \{1\}$ where $F^\times$ is taken with the diagonal embedding, we have $\mathbb{I}_{F,S} \cap (F^\times \mathbb{U}_{F,S}) = E_{F,S}$, hence we have an inclusion $\mathbb{U}_{F,S} \hookrightarrow C_F$ and the inclusion $\mathbb{I}_{F,S} \hookrightarrow \mathbb{I}_F$ passes to the quotient by $E_{F,S}$, so we have an inclusion $C_{F,S} \hookrightarrow C_F/\mathbb{U}_{F,S}$. Hence the cokernel is

$$\mathbb{I}_F/(F^\times \mathbb{U}_{F,S}\mathbb{I}_{F,S}) \cong (\bigoplus_{v \notin S} \mathbb{Z})/F^\times =: Cl_{F,S}$$

□

Remark 1.2.4. If S is the complementary of finitely many places, then $Cl_{F,S} = 1$ for the Chinese reminder theorem (it is a Dedekind domain with finitely many prime ideals) and $C_{F,S} \xrightarrow{\sim} C_S(F)$ is an isomorphism.

Remark 1.2.5. Since (G, C_F) is a class formation, if $H_S = G_{K_S}$, then $(G_S, C_F^{H_S})$ is a P-class formation (P is constructed to do so).

Proposition 1.2.6. *There is a canonical exact sequence*

$$0 \to \mathbb{U}_S \to C_K^{H_S} \to C_S \to 0$$

Proof. Remark that since for Hilbert 90 $H^1(G(F_1/F), F_1^\times) = 0$, we have for each finite extension F_1/F the exact sequence

$$0 \to F_1^\times \to \mathbb{I}_{F_1} \to C_{F_1} \to 0$$

and since $(F_1^\times)^{G(F_1/F)} = F^\times$ and $\mathbb{I}_{F_1}^{G(F_1/F)} = \mathbb{I}_F$, then $C_F = C_{F_1}^{G(F_1/F)}$, so in particular

$$\varinjlim_{K_S/F/K} C_F = C_K^{H_S}$$

If S is the complement of finitely many places, by previous lemma we have an isomorphism $C_{F,S} \cong C_F/\mathbb{U}_{F,S}$, so passing to the filtered colimit over F we have $C_S \cong C_K^{H_S}/U_S$, which gives the exact sequence.
In general, to reduce to this case we need to show that $\varinjlim_{K_S/F/K} Cl_{F,S} = 0$.
Consider L/F the maximal unramified extension of K such that every non-archimedean place of S splits completely, and consider F' the maximal abelian subextension of L/F. Then F'/F is the maximal abelian extension of F which splits completely on the primes of S, so $F \subseteq H_F$ where H_F is the Hilbert class field, so F'/F is finite.
By class field theory ([CF67, XII]) and since $Gal(L/F)^{ab} = Gal(F'/F)$ since the commutator

$$[Gal(L/F), Gal(L/F)] = Gal(L/F')$$

We have a commutative diagram

$$\begin{tikzcd} Cl_{F,S} \arrow[r, "\sim"] \arrow[d] & Gal(L/F)^{ab} \arrow[r, equal] \arrow[d, "V"] & Gal(F'/F) \\ Cl_{F',S} \arrow[r, "\sim"] & Gal(L/F')^{ab} & \end{tikzcd}$$

Where V is the transfer map (see [AT67, XIII,2]). Then we have a theorem

Theorem 1.2.7 (Principal ideal theorem)**.** *Let U be a group whose commutator $[U,U]$ is of finite index and finitely generated. Then the transfer map*

$$V : U/[U,U] \to [U,U]/[[U,U],[U,U]]$$

is zero

Proof. [AT67, XIII,4]

Since here $U = Gal(L/F)$ and $[U,U] = Gal(L/F')$, $[U : [U,U]] = [F' : F] = \#Cl(F) < \infty$. □

Lemma 1.2.8. *$H^r(G_S, \mathbb{U}_s) = 0$ for $r \geq 1$*

Proof. By definition

$$H^r(G_S, \mathbb{U}_S) = \varinjlim_F H^r(Gal(F/K), \prod_{w \notin S_F} \mathcal{O}_w^\times)$$

And since $Gal(F/K)$ is a finite group, we can take out the product[3] and

$$\varinjlim_F \prod_{v \notin S_K} (Gal(F/K), \prod_{w|v} \mathcal{O}_w^\times)$$

And since for all v $Gal(F/K) = \prod_{w|v} Gal(F_w/K_v)$ and the only factor that acts on $\mathcal{O}_w^\times$ is $Gal(F_w/K_v)$, we have

$$\prod_{v \notin S_K} H^r(Gal(F/K), \prod_{w|v} \mathcal{O}_w^\times) = \prod_{\substack{v \notin S_K \\ w|v}} H^r(Gal(F_w/K_v), \mathcal{O}_w^\times)$$

Then since $w|v$ is unramified, $\mathcal{O}_w^\times \times \pi^{\mathbb{Z}} \cong F_w^\times$, and for the valuation exact sequence

$$0 \to \mathcal{O}_v = (\mathcal{O}_w)^{Gal(F_w/K_v)} \to K_v^\times = (F_w^\times)^{Gal(F_w/K_v)} \to \pi^{\mathbb{Z}} = (\pi^{\mathbb{Z}})^{Gal(F_w/K_v)} \to 0$$

$H^1(Gal(F_w/K_v), \mathcal{O}_w) = 0$. And since it is unramified, $Gal(F_w/K_v)$ is finite cyclic, so it is enough to prove that $\widehat{H}^0(Gal(F_w/K_v), \mathcal{O}_w^\times) = 0$, and this is true since $N_{F_w/K_v} : \mathcal{O}_w \to \mathcal{O}_v$ is surjective □

Corollary 1.2.9. *The exact sequence of proposition 1.2.6 is again exact applying $(\_)^{G_S}$, since $C_{\overline{K}}^{G_S} = C_K$ and $\mathbb{U}_S^{G_S} = \mathbb{U}_{K,S}$, we have $C_S^{G_S}/\mathbb{U}_S^{G_S} = C_S(K)$ so it gives isomorphisms*

$$C_S(K) \xrightarrow{\sim} C_S^{G_S} \tag{1.1}$$

$$H^r(G_S, C_{\overline{F}}^{H_S}) \xrightarrow{\sim} H^r(G_S, C_S) \tag{1.2}$$

In particular (G_S, C_S) is a P class formation.

[3]If G is a discrete group, $\{M_i\}_I$ a family of G-modules, then

$$H^r(G, \prod_I M_i) \cong \mathrm{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}, \prod_I M_i) \cong \prod_I \mathrm{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}, M_i) \cong \prod_I H^r(G, M_i)$$

Definition 1.2.10. We denote $D_S(F)$ and D_F the connected components of $C_S(F)$ and C_F.

Remark 1.2.11. If F is a function field, then since every nonarchimedean field is totally disconnected $D_S(F) = D_F = \{1\}$. If F is a number fields, then one has that $D_S(K)$ is the closure of the image of D_K in $C_S(K)$, and since $\mathbb{U}_{S,K}$ is compact the map is closed, so $D_S(F) = D_F U_{S,F}/U_{S,F}$.

Lemma 1.2.12. *If K is a number field, then $D_S(K)$ is divisible and there is an exact sequence*

$$0 \to D_S(K) \to C_S(F) \xrightarrow{rec} G_S^{ab} \to 0$$

Proof. If S contains all the primes, then the exact sequence is the reciprocity law of global class field theory (see [AT67]), and D_K is divisible since

$$D_K = \mathbb{R} \times (\mathbb{R}/\mathbb{Z})^s \times \mathbb{S}^{r+s-1}$$

where r and $2s$ are the real and complex embeddings of K and $\mathbb{S} \cong (\mathbb{R} \times \widehat{\mathbb{Z}})/\mathbb{Z}$ is the solenoid, which are all divisible groups.
In the general case, $D_S(K)$ is divisible since D_K is, and quotients of divisible are divisible.
The image of $\mathbb{U}_{S,K}$ in G^{ab} is the subgroup fixing $K^{ab} \cap K_S$, hence it is the kernel of $G^{ab} \to G_S^{ab}$. So we have a commutative diagram

$$\begin{tikzcd}
 & \mathbb{U}_{K,S} \arrow[r] \arrow[d] & Gal(K^{ab}/K^{ab} \cap K_S) \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & D_K \arrow[r] \arrow[d, two heads] & C_K \arrow[r] \arrow[d, two heads] & G^{ab} \arrow[r] \arrow[d, two heads] & 0 \\
 & D_{K,S} \arrow[r] & C_{K,S} \arrow[r] & G_S^{ab} \arrow[r] & 0
\end{tikzcd}$$

and we conclude by snake lemma. □

Remark 1.2.13. This says that the reiprocity map induces $\alpha^0 = {}_n rec(G_S, \mathbb{Z}/n\mathbb{Z})$ and $\alpha^1 = rec(G_S, \mathbb{Z}/n\mathbb{Z})_n$, which are respectively epi and iso if $n = \ell^m$ with $\ell \in P$

Theorem 1.2.14. *Let M be a finitely generated G_S-module and $\ell \in P$.*

(a) The map

$$\alpha^r(G_S, M) : Ext^r_{G_S}(M, C_S) \to H^{2-r}(G_S, M)^*$$

is an isomorphism for all $r \geq 1$

(b) If K is a function field, then there is an isomorphism

$$Hom_{G_S}(M, C_S)^\wedge \xrightarrow{\sim} H^2(G_S, M)^*$$

Where $\wedge$ is the profinite completion, *Proof: K Number field.* We have that $\alpha^1(G_S, \mathbb{Z}/\ell^m\mathbb{Z})$ is iso and $\alpha^0(G_S, \mathbb{Z}/\ell^m\mathbb{Z})$ is epi, so (a) follows from theorem 1.2.2, and also $\alpha^0(G_S, M)(\ell)$ is epi if M is finite.

□

Proof: K Function field. If K is a function field, then P contains all primes and $rec : C_K \to G^{ab}$ is injective with dense image, and there is an exact sequence

$$0 \to C_K \to G^{ab} \to \widehat{\mathbb{Z}}/\mathbb{Z} \to 0$$

Using the same argument as in lemma 1.2.12, we have the exact sequence

$$0 \to C_S(K) \xrightarrow{rec} G_S^{ab} \to \widehat{\mathbb{Z}}/\mathbb{Z} \to 0$$

And since $\widehat{\mathbb{Z}}/\mathbb{Z}$ is uniquely divisibile, $\alpha^0(G_S, \mathbb{Z}/\ell^m\mathbb{Z}) = {}_{\ell^n}rec$ and $\alpha^0(G_S, \mathbb{Z}/\ell^m\mathbb{Z}) = rec_{\ell^n}$ are isomorphisms. □

1.3 Tate-Poitou

Let us fix M a finitely generated G_S-module whose order of the torsion group is a unit in $\mathcal{O}_S$.
Let v be a place of K and choose an embedding $K \hookrightarrow K_v$, $G_v = Gal(\overline{K}_v/K_v)$ the decomposition subgroup and if v is nonarchimedean, let $k(v)$ be the residue field and $g_v = Gal(\overline{k(v)}/k(v)) = G_v/I_v$. The embedding gives a canonical map $G_v \to G_K$ which induces by the quotient a canonical map $G_v \to G_S$, which gives a map

$$H^r(G_S, M) \to H^r(G_v, M)$$

We will consider

$$H^r(K_v, M) = \begin{cases} H^r(G_v, M) & \text{if } v \text{ is non archimedean} \\ \widehat{H}^r(G_v, M) & \text{if } v \text{ is archimedean} \end{cases}$$

In particular $H^0(\mathbb{R}, M) = M^{Gal(\mathbb{C}/\mathbb{R})}/N_{\mathbb{C}/\mathbb{R}}M$ and $H^0(\mathbb{C}, M) = 0$.
If v is non archimedean and M is unramified (i.e. $M^I = M$), we have a canonical map $H^r(g_v, M) \to H^r(G_v, M)$ and we will write $H^r_{un}(K_v, M)$ the image of this map, so by definition $H^0_{un}(K_v, M) = H^0(K_v, M)$ and for the inflation-restriction exact sequence $H^1_{un}(K_v, M) = H^1(g_v, M)$, and if M is torsion since $cd(g_v) \leq 1$ we have $H^r_{un}(K_v, M) = 0$ for $r > 1$.
Since a finitely generated G_S-module is ramified only on finitely many places of S, we can define

$$P^r_S(K, M) = \prod_{v \in S}{}^{H^r_{un}(K_v,M)} H^r(K_v, M)$$

with the restricted product topology.

Lemma 1.3.1. *The image of*

$$H^r(G_S, M) \to \prod_{v \in S} H^r(K_v, M)$$

is contained in $P^r_S(K, M)$

Proof. If $\gamma \in H^r(G_S, M)$, since $H^r(G_S, M) = \varinjlim H^r(G_S/U, M^U)$, then there is a finite extension $K_S/L/K$ such that $\gamma \in H^r(Gal(L/K), M)$. So since if $w|v$ is unramified $Gal(L_w/K_v) = Gal(k(w)/k(v))$, we have $\gamma \in H^2_{un}(K_v, M)$. □

So we have a map $\beta^r : H^r(G_S, M) \to P^r_S(K, M)$

Lemma 1.3.2. *If M is finite, the inverse image of every compact subset of $P^1(G_S, M)$ under β^1 is finite.*

Proof. Consider $U = Gal(L/K) \subseteq G_S$ open normal such that $M^U = M$, then

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(U,M) & \longrightarrow & H^1(G_S,M) & \longrightarrow & H^1(G/U,M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^1(L,M) & \longrightarrow & P^1(K,M) & \longrightarrow & \prod' H^1(G/U,M) & \longrightarrow & 0
\end{array}$$

Since $H^1(G/U, M)$ is finite, every subset is finite. So it is enough to prove it if G_S acts trivially on M.
For every V compact neighborhood of 1 there exists $T \subseteq S$ such that $S \setminus T$ is finite and V is contained in

$$P(T) = \prod_{v \in S \setminus T} H^1(K_v, M) \times \prod_{v \in S} H^1_{un}(K_v, M)$$

So it is enough to show that the inverse image of $P(T)$ is finite. Let $f \in (\beta^1)^{-1}(P(T)) \subseteq H^1(G_S, M) = \mathrm{Hom}_{Grp}(G_S, M)$, then f is by defintion such that $K_S^{ker(f)}$ is unramified at all places $v \in T$. So since $[K_S^{ker(f)} : K] = \#(G_S/Ker(f))$, we have $[K_S^{ker(f)} : K]$ divides $\#M$, and it is unramified outside the finite set $S \setminus T$. Hence by Hermite's theorem there are only finitely many extension like this, so $(\beta^1)^{-1}(P(T))$ is finite. □

We define $Ш^r_S = Ker(\beta^r)$. In particular, if M is a finite G-module, since $P^1_S(K, M)$ is locally compact $Ш^1_S$ is finite.

Remark 1.3.3. If M is a finite G_S-module, then $M^D = \mathrm{Hom}(M, K_S^\times) = \mathrm{Hom}(M, E_S)$ is again a finite G_S-module and if $\#M$ is invertible in $\mathcal{O}_{K,S}$, then $M^D = \mathrm{Hom}(M, \overline{K}^\times) = \mathrm{Hom}(M, \mu_\infty(\overline{K})) = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$, so M^{DD} is canonically isomorphic to M for Pontryagin duality. So by the local results we can conclude that

$$P^r_S(K, M) \cong P^{2-r}_S(K, M^D)^*$$

is a topological isomorphism. Then, by taking the dual map of β^{2-r}, we have continuous maps

$$\gamma^r(K, M^D) : P^r_S(K, M^D) \to H^{2-r}(G_S, M)^*$$

Theorem 1.3.4. *Let M be a finite G_S-module whose order is a unit in $\mathcal{O}_{K,S}$. Then:*

1. *The map $\beta^0_S(K,M)$ is injective and $\gamma^2(K,M^D)$ is surjective, for $r = 0,1,2$ we have isomorphisms:*

$$Im(\beta^r) = Ker(\gamma^r)$$

such that there is an exact sequence of locally compact groups:

$$\begin{tikzcd}
0 \arrow[r] & H^0(G_S,M) \arrow[r, "\beta^0"] & P^0_S(K,M) \arrow[r, "\gamma^0"] & H^2(G_S,M^D)^* \arrow[d] & \\
& H^1(G_S,M^D)^* \arrow[d] & P^1_S(K,M) \arrow[l, "\gamma^1"'] & H^1(G_S,M) \arrow[l, "\beta^1"'] & \\
& H^2(G_S,M) \arrow[r, "\beta^2"] & P^2_S(K,M) \arrow[r, "\gamma^2"] & H^0(G_S,M^D)^* \arrow[r] & 0
\end{tikzcd}$$

(Tate-Poitou exact sequence 1.3)

with the following topological description:

finite	*compact*	*compact*
compact	*locally compact*	*discrete*
discrete	*discrete*	*finite*

2. *For $r \geq 3$, β^r is a bijection*

$$H^r(G_S,M) \to \prod_{v\ real} H^r(K_v,M)$$

In particular they are all finite.

1.3.1 Proof of the main theorem

Let M be a finitely generated G_S-module. We will define with the same notation M^d three different objects:

- When M is regarded as a G_S module, then $M^d = \mathrm{Hom}_{G_S}(M,E_S)$
- If M is not ramified on v, M can be regarded as a g_v-module, so in this case $M^d = \mathrm{Hom}_{g_v}(M,\mathcal{O}_v^{un\times})$, where $\mathcal{O}_v^{un}$ is the ring of integers of the maximal unramified extension of K_v.
- When M is regarded as a G_v module, $M^d = \mathrm{Hom}_{G_v}(M,\overline{K_v}^\times)$

Lemma 1.3.5. *Let M be a finitely generated G_S-module where $\#M_{tor}$ is a unit in $\mathcal{O}_{K,S}$. Then*

(a) For all $r \geq 0$, $Ext^r_{G_S}(M,E_S) = H^r(G_S,M^d)$

(b) For $v \notin S$, $H^r(g_v,M^d) = Ext^r_{g_v}(M,\mathcal{O}_v^{un\times})$ and for $r \geq 2$ they are both zero.

Proof. (a) Since E_S is divisible by all integers that are units in $\mathcal{O}_{K,S}$, we have $\mathcal{E}xt^r_{\mathbb{Z}}(M, E_S) = \mathrm{Ext}^r_{\mathbb{Z}}(M, E_S) = 0$ for $r \geq 1$, hence the result comes from the degenerating spectral sequence

$$H^p(G_S, \mathcal{E}xt^q_{\mathbb{Z}}(M, E_S)) \Rightarrow \mathrm{Ext}^{p+q}_{G_S}(M, E_S)$$

(b) Since again $\mathcal{O}_v^{un\times}$ is divisible by all the integers dividing $\#M_{tor}$, the equality comes again from the spectral sequence.
Now $\mathcal{O}_v^{un\times}$ is cohomologically trivial, so for a well-known theorem ([Ser62, IX, Âğ7, TheorÃÍme 11]) we have an injective resolution

$$0 \to \mathcal{O}_v^{un\times} \to I^1 \to I^2 \to 0$$

In particular $\mathrm{Ext}^r_{g_v}(M, \mathcal{O}_v^{un\times}) = 0$ for all M and all $r \geq 2$.

□

Lemma 1.3.6. *If now either M is finite or S omits finitely many places, then*

$$Hom_{G_S}(M, \mathbb{I}_S) = \prod_{v \in S} H^0(G_v, M^d)$$

(which is $P^0_S(K, M)$ if K is a function field, since K_v is always non archimedean and we don't have Tate cohomology groups involved), and for $r \geq 1$

$$Ext^r_{G_S}(M, \mathbb{I}_S) = P^r_S(K, M^d)$$

Proof. Consider $T \subsetneq S$ finite such that T contains all the archimedean places and all the nonarchimedean places where M is ramified (they are finitely many), and the order of M_{tors} is invertible in $\mathcal{O}_{K,T}$. Consider the subgroup of the idÃÍle group $\mathbb{I}_{F,S}$:

$$\mathbb{I}_{F,S\supseteq T} := \prod_{w \in T} F_w^\times \times \prod_{w \in S \setminus T} \mathcal{O}_v^\times$$

Then

$$\mathbb{I}_S = \varinjlim_{\substack{T \subseteq S \\ \text{opportune}}} \varinjlim_{F \subseteq K_T} \mathbb{I}_{F,S\supseteq T}$$

So in particular

$$\mathrm{Ext}^r_{G_S}(M, \mathbb{I}_S) = \varinjlim_{F,T} \mathrm{Ext}^r_{Gal(F/K)}(M, \mathbb{I}_{F,S\supseteq T})$$

And since Ext commute with the products, for Shaphiro's Lemma:

$$\mathrm{Ext}^r_{Gal(F/K)}(M, \mathbb{I}_{F,S\supseteq T}) = \prod_{v \in T} \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F_w^\times) \times \prod_{v \in S \setminus T} \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, \mathcal{O}_{F_w}^\times)$$

Consider F big enough such that $K_v^{un} \subseteq F_w$. Since now if I_w is the inertia group of $Gal(F_w/K_v)$, we have $H^r(I_w, \mathcal{O}_{F_w}^\times) = 0$ so the degenerating spectral sequence

$$\mathrm{Ext}^p_{g_v}(M, H^q(I_w, \mathcal{O}_{F_w}^\times)) \Rightarrow \mathrm{Ext}^{p+q}_{Gal(F_w/K_v)}(M, \mathcal{O}_{F_w}^\times)$$

gives the isomorphism

$$\mathrm{Ext}^p_{Gal(F_w/K_v)}(M, \mathcal{O}^\times_{F_w}) = \mathrm{Ext}^p_{g_v}(M, \mathcal{O}^{un\times}_v) = H^p(g_v, M^d)$$

Hence combining this we have

$$\mathrm{Ext}^r_{Gal(F/K)}(M, \mathbb{I}_{F,S\supseteq T}) = \prod_{w\in T} \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F^\times_w) \times \prod_{w\in S\setminus T} H^r(g_v, M^d)$$

Considering now F/K finite such that G_{F_w} acts trivially on M, so

$$\mathrm{Hom}_{Gal(F_w/K_v)}(M, F^\times_w)) = \mathrm{Hom}_{G_v}(M, \overline{K}^\times_v)$$

and looking at the spectral sequence

$$\mathrm{Ext}^p_{Gal(F_w/K_v)}(M, H^q(G_{F_w}, \overline{K}^\times_v)) \Rightarrow \mathrm{Ext}^{p+q}_{G_v}(M, \overline{K}^\times_v)$$

the five-term exact sequence and Hilbert 90 ($H^1(G_{F_w}, \overline{K}^\times_v) = 0$) give

$$\mathrm{Ext}^1_{Gal(F_w/K_v)}(M, F^\times_w)) = \mathrm{Ext}^1_{G_v}(M, \overline{K}^\times_v)$$

So for $r = 0, 1$, since $\mathrm{Ext}^r_{G_v}(M, \overline{K}^\times_v) = H^r(G_v, M^d)$ we have

$$\mathrm{Ext}^r_{G_s}(M, \mathbb{I}_S) = \varinjlim(\prod_{v\in T} H^r(G_v, M^d) \times \prod_{v\in S\setminus T} H^r(g_v, M^d))$$

which gives the result for $r = 0, 1$.
For $r \geq 2$, $H^r(g_v, M^d) = 0$, so since T is finite we have

$$\mathrm{Ext}^r_{G_S}(M, \mathbb{I}_S) = \varinjlim_{\overline{K}/F/K} (\bigoplus_{v\in S} \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F^\times_w)) = \bigoplus_{v\in S}(\varinjlim_{\overline{K}/F/K} \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F^\times_w))$$

If v is archimedean, it is trivial that $\mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F^\times_w) = \mathrm{Ext}^r_{G_v)}(M, \overline{K_v}^\times) = H^r(G_v, M^d)$, so suppose now v non archimedean.

Claim *If S contains almost all primes, $\varinjlim_{K_S/F/K} F_w = \overline{K_v}$.*
If we fix an extension $K_v \subseteq L_w$ of degree n generated by f_w, for all the places $u \notin S$ there is a unique unramified extension L_u/K_u of degree n generated by the root of a polynomial $f_u \in K_u[X]$, and the weak approximation theorem gives $f \in K[X]$ such that $|f - f_u|_u < \epsilon$ for all $u \notin S$ and for $u = v$, and for Krasner's lemma if F is generated by a root of f then there exists $u'|u$ such that $F_{u'} = L_u$ for all $u \notin S$ and for $u = w$, i.e. for all $v \in S$ and all L/K_v finite separable there exists $K_S/F/K$ such that $L = F_w$ for $w|v$. So in this case we conclude

since

$$\varinjlim_F \mathrm{Ext}^r_{Gal(F_w/K_v)}(M, F^\times_w)) = \mathrm{Ext}^r_{G_v}(M, \overline{K_v}^\times)) = H^r(G_v, M^d)$$

Claim *If M is finite and ℓ divides the order of M, then for all $v \in S$ finite and for all N there is $K_S/F/K$ such that $\ell^N | [F_w : K_v]$.*
If $K = k(X)$ is a function field, this is trivial: it is enough to take the unique extension of k of degree ℓ^N.
If K is a number field, then S contains all the places over ℓ since ℓ is a unit in $\mathcal{O}_{K,S} = \bigcap_{v \notin S} \mathcal{O}_v$, and for all n consider $F = K(\zeta_{\ell^n}) \subseteq K_S$, so $K_v(\zeta_{\ell^n}) \subseteq F_w$, so let p be the rational prime such that $v|p$, we have a diamond

$$\begin{tikzcd}[column sep=small] & K_v(\zeta_{\ell^n}) \arrow[dl, dash] \arrow[dr, dash] & \\ K_v \arrow[dr, dash] & & \mathbb{Q}_p(\zeta_{\ell^n}) \arrow[dl, dash] \\ & \mathbb{Q}_p & \end{tikzcd}$$

So if $\ell \neq p$ $\mathbb{Q}_p(\zeta_{\ell^n})$ is unramified over $\mathbb{Q}_p$ of degree d dividing $\ell^{n-1}(\ell - 1)$, and $d \to \infty$ if $n \to \infty$, and if $\ell = p$ $\mathbb{Q}_p(\zeta_{\ell^n})$ is totally ramified of degree $\ell^{n-1}(\ell - 1)$ so for $n >> 0$ since $[K_v : \mathbb{Q}_p]$ is fixed $\ell^N | [F_w : K_v]$.
So if v is non archimedean we have

$$H^r(G_{F_w}, \overline{K_v}^{\times}) = \begin{cases} F_w^* & \text{if } r = 0 \\ \mathbb{Q}/\mathbb{Z} & \text{if } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

And by the claim:

$$\varinjlim_F H^2(G_{F_w}, \overline{K_v}^{\times})(\ell) = \varinjlim_F (\mathbb{Q}/\mathbb{Z}(\ell) \xrightarrow{[F_w:K_v]} \mathbb{Q}/\mathbb{Z}(\ell)) = 0$$

And since the ℓ-primary component of an abelian group A is $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_\ell, A)$ and it is a morphism of $Gal(F_w/K_v)$-modules since the action of $Gal(F_w/K_v)$ must respect the order of the elements.
Since now if M is finite we have

$$\mathrm{Hom}_{Gal(F_w/K_v)}(M, \_) \cong \bigoplus_{\ell | \# M} \mathrm{Hom}_{Gal(F_w/K_v)}(M, (\_)(\ell))$$

And since $\mathbb{Q}/\mathbb{Z}$ is divisible, $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_\ell, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_\ell, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}(\ell)$ and so there is a quasi isomorphism

$$R\mathrm{Hom}_{Gal(F_w/K_v)}(M, \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{\ell | \# M} R\mathrm{Hom}_{Gal(F_w/K_v)}(M, \mathbb{Q}/\mathbb{Z}(\ell))$$

Hence we have a direct system of spectral sequences

$$\mathrm{Ext}^p_{Gal(F_w/K_v)}(M, H^q(G_{F_w}, \overline{K_v}^{\times})) \Rightarrow \mathrm{Ext}^{p+q}_{G_v}(M, \overline{K_v}^{\times})$$

which shows that

$$\varinjlim_{F} \operatorname{Ext}^r_{Gal(F_w/K_v)}(M, F_w^\times) = \operatorname{Ext}^r_{G_v}(M, \overline{K_v}^\times) = H^r(G_v, M^d)$$

and this concludes the proof. $\square$

Conclusion. So assuming now that M is finite, we have the long exact sequence from the triangle given by lemma 1.2.3 and by definition of $C_{F,S}$:

$$\cdots \to \operatorname{Ext}^r_{G_S}(M^D, E_S) \to \operatorname{Ext}^r_{G_S}(M^D, \mathbb{I}_S) \to \operatorname{Ext}^r_{G_S}(M^D, C_S) \to \cdots$$

Recall that γ^2 is the dual of $H^0(G_S, M) \to P^0_S(K, M)$, which is mono, so it is epi, hence by the previous lemmas we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(G_S, M) \arrow[r] & \prod_{v\in S} H^0(K_v, M) \arrow[r] & \operatorname{Hom}_{G_S}(M^D, C_S) \arrow[d] \\
& H^1(G_S, M^D)^* \arrow[d] & P^1_S(K, M) \arrow[l, "\gamma^1"'] & H^1(G_S, M) \arrow[l, "\beta^1"'] \\
& H^2(G_S, M) \arrow[r, "\beta^2"] & P^2_S(K, M) \arrow[r, "\gamma^2"] & H^0(G_S, M^D)^* \arrow[r] & 0
\end{tikzcd}$$

And for $r \geq 3$ we have $H^r(G_S, M) \cong \bigoplus_{v \text{ real}} H^r(G_v, M)$ So if K is a function field $P^0_S(K, M) = \prod_{v\in S} H^0(K_v, M)$ and for theorem 1.2.14, part (b), $\operatorname{Hom}_{G_S}(M^D, C_S)^\wedge \cong H^2(G_S, M^D)^*$, and since M^D is finite $H^2(G_S, M^D)^*$ is finite, hence $\operatorname{Hom}_{G_S}(M^D, C_S)$ is complete, so we conclude. If K is a number field, consider that exact sequence for the finite module M^D:

$$\begin{tikzcd}
H^1(G_S, M)^* \arrow[d] & P^1_S(K, M^D) \arrow[l, "\gamma^1"'] \\
H^2(G_S, M^D) \arrow[r, "\beta^2"] & P^2_S(K, M^D) \arrow[r, "\gamma^2"] & H^0(G_S, M)^* \arrow[r] & 0
\end{tikzcd}$$

and by dualizing it

$$\begin{tikzcd}
H^1(G_S, M) \arrow[r, "\beta^1"] & P^1_S(K, M) \\
H^2(G_S, M^D)^* \arrow[u] & P^0_S(K, M) \arrow[l, "\gamma^0"'] & H^0(G_S, M) \arrow[l, "\beta^0"'] & 0 \arrow[l]
\end{tikzcd}$$

So we conclude. $\square$

Corollary 1.3.7. *There is a canonical perfect pairing of finite groups*

$$Ш^1_S(K, M) \times Ш^2_S(K, M^D) \to \mathbb{Q}/\mathbb{Z}$$

In particular $Ш^2_S(K, M)$ *is finite.*

Proof. By definition $Ш^2_S(K, M^D) = ker(\beta^2 : H^2(G_S, M^D) \to P^2_S(K, M^D))$, so since we have $\beta^2(K, M^D)^* = \gamma^0(K, M)$ So by the main theorem:

$$Ш^2_S(K, M^D)^* = coker(\gamma^0) \cong ker(\beta^1) = Ш^1_S(K, M^D)$$

Corollary 1.3.8. *With the same hypothesis of the theorem, if S is finite then $H^r(G_S, M)$ is finite.*

Proof. $P^r_S(K, M)$ is finite in this case because if $v \in S_f$ then $H^r(G_v, M)$ is finite for local Tate duality, and if $v \in S_\infty$ $H^r_T(G_v, M)$ is finite since G_v is finite, so $H^0(G_S, M)$ is finite, and $H^1(G_S, M)$ and $H^2(G_S, M)$ are finite because $Ш^1_S(K, M)$ and $Ш^2_S(K, M)$ are.

Chapter 2

Proper Base Change

The aim of this chapter will be to prove the following theorem:

Theorem 2.0.1. *Let $X \xrightarrow{f} Y$ be a proper morphism of schemes. Let $Y' \xrightarrow{g} Y$ be a morphism of schemes. Set $X' = X \times_Y Y'$ and consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then for any F torsion sheaf on X_{et} the canonical morphism gives an isomorphism of sheaves over Y'

$$g^* R^p f_* F \cong R^p f'_* g'^* F$$

We notice that we have the following:

Corollary 2.0.2. *Let $X \xrightarrow{f} S$ a proper morphism and F an abelian torsion sheaf over X, and let $s \to S$ be a geometric point, X_s the fiber $X \times_S Spec(k(s))$. Then, $\forall\ q \geq 0$ we have*

$$(R^q f_* F)_s \cong H^q(X_s, F)$$

Proof. Take $Y' = s$

Corollary 2.0.3. *Let $(A, \mathfrak{M}, k)$ be a strictly local ring, $S = Spec(A)$, $X \xrightarrow{f} S$ a proper morphism, X_0 the closed fiber of f (i.e. the fiber over the only closed point, i.e. $X \times_S Spec(k)$). Then $\forall\ q \geq 0$ we have*

$$H^q(X, F) \cong H^q(X_0, F)$$

Proof. Follows from the previous corollary and because $(R^q f_* F)_{\bar{s}} = H^q(X, F)$ since A is strictly local.

Proposition 2.0.4. *Corollary 2.0.3 implies theorem 2.0.1*

Proof. Since being equal is a local property, we can suppose $Y = Spec(A)$ and $Y' = Spec(A')$ affine, and by passing to the limit we can suppose Y' of finite type over Y.
Consider $y' \in Y'$ a geometric point closed in $g^{-1}g(y')$. Then the theorem is true if and only if

$$(g^* R^p f_* F)_{\bar{y}'} \cong (R^p f'_* g'^* F)_{\bar{y}'}$$

We have

$$\begin{aligned}(g^* R^p f_* F)_{\bar{y}'} = (R^p f_* F)_{g(\bar{y}')} &= H^p(X \times_Y Spec(\mathcal{O}^{sh}_{Y,g(y')}), F)\\ (R^p f'_* g'^* F)_{\bar{y}'} &= H^p(X \times_Y Spec(\mathcal{O}^{sh}_{Y',y'}), F)\end{aligned}$$

Applying corollary 2.0.3 to $X \times_Y Spec(\mathcal{O}^{sh}_{Y,g(y')} \to Spec(\mathcal{O}^{sh}_{Y,g(y')}$ and $X \times_Y Spec(\mathcal{O}^{sh}_{Y',y'} \to Spec(\mathcal{O}^{sh}_{Y',y'}$ we have

$$\begin{aligned}H^p(X \times_Y Spec(\mathcal{O}^{sh}_{Y,g(y')}), F) &= H^p(X \times_Y Spec(\overline{k(g(y'))}), F)\\ (R^p f'_* g'^* F)_{\bar{y}'} &= H^p(X \times_Y Spec(\overline{k(y')}), F)\end{aligned}$$

By hypothesis, $k(y')$ is algebraic over $k(g(y'))$ since it is closed in $g^{-1}g(y')$, so $\overline{k(y')} \cong \overline{k(g(y'))}$ $\square$

So in the rest of the chapter I will prove corollary 2.0.3 only in the context where A is noetherian: this will imply theorem 2.0.1 when Y and Y' are locally noetherian.

2.1 Step 1

Throughout this section, I will prove the proper base change for $q = 0$ or 1, and $F = \mathbb{Z}/n\mathbb{Z}$.
For $q = 0$, the theorem follows from:

Proposition 2.1.1 (Zariski Connection Theorem). *Let $(A, \mathfrak{m})$ be an henselian Noetherian ring and $S = Spec(A)$. Let $X \xrightarrow{f} S$ a proper morphism, X_0 the closed fiber. Then we have a bijection between the connected components of X and of X_0*

Proof. Since X and X_0 are noetherian schemes, we have that the connected components are all and only the open and closed subsets, which are in bijection with the idempotents of $\Gamma(X, \mathcal{O}_X)$. So the goal is to show that the canonical map

$$Idem\Gamma(X, \mathcal{O}_X) \longrightarrow Idem\Gamma(X_0, \mathcal{O}_{X_0})$$

is bijective.
We have the following lemma:

Theorem. *Let $X \to Y$ a proper morphism, Y a Noetherian scheme. Then $\forall$ F coherent $\mathcal{O}_X$-modules $R^p f_* F$ is a coherent $\mathcal{O}_Y$-module*

Proof. [GD61, III.3.2.1]

It follows that $\Gamma(X, \mathcal{O}_X)$ is a finite A-algebra.
For all n one consider the formal completion of X with respect to $\mathfrak{m}$

$$X_n = X \times_A Spec(A/\mathfrak{m}^{n+1})$$

In particular $\Gamma\widehat{(X, \mathcal{O}_X)} = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the $\mathfrak{m}$-adic completion.
Recall that A is Henselian if and only if all finite A-algebras are finite products of local rings ([Sta, Tag 04GG]), whose only idempotents are uples with 0 and 1, hence the canonical map

$$Idem(\Gamma(X, \mathcal{O}_X)) \to Idem(\Gamma\widehat{(X, \mathcal{O}_X)})$$

is bijective.
We have again the following theorem:

Theorem. *If $X \xrightarrow{f} Y$ is a proper morphism of Noetherian schemes Y' a closed subscheme of Y, $X' = f^{-1}Y'$ defined by the sheaf of $\mathcal{O}_X$-ideals $\mathcal{J}$, $\hat{Y}$ the formal completion of Y with respect to Y', $\hat{X}$ the formal completion of X with respect to X', $\hat{f}$ the natural map induced on the completions, $F_k = F/\mathcal{J}^{k+1}F$, $\hat{F}$ the extension of F to $\hat{X}$, then*

- *$R^p\hat{f}_*\hat{F}$ is a coherent $\mathcal{O}_{\hat{X}}$ module.*
- *$\forall$ n there is a commutative diagram*

$$\begin{tikzcd} \widehat{R^n f_* F} \arrow[r, "\rho_n"] \arrow[d, "\phi_n"] & R^n\hat{f}_*(\hat{F}) \arrow[dl, "\psi_n"] \\ \varprojlim_k R^n f_* F_k & \end{tikzcd}$$

and ρ_n, ϕ_n and ψ_n are topological isomorphisms $\forall$ n

Proof. [GD61, III.4.1]

In particular the canonical map

$$\Gamma\widehat{(X, \mathcal{O}_X)} \to \varprojlim \Gamma(X_k, \mathcal{O}_{X_k})$$

is an isomorphism, hence

$$Idem(\Gamma(X, \mathcal{O}_X)) \to \varprojlim Idem(\Gamma(X_k, \mathcal{O}_{X_k}))$$

Is bijective. Since X_k and X_0 have the same underlying topological space, we have that

$$Idem(\Gamma(X_k, \mathcal{O}_{X_k})) \to Idem(\Gamma(X_0, \mathcal{O}_{X_0}))$$

is bijective $\forall$ k, so we conclude. □

In order to conclude for $q = 1$ and $F = \mathbb{Z}/n\mathbb{Z}$, recall that

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) = \{\text{n-torsors over X}\} = \{\text{finite Ãľtale coverings with group } \mathbb{Z}/n\mathbb{Z}\}$$

So the proof for $q = 1$ is given by

Proposition 2.1.2. *Let A be a local henselian noetherian ring, $S = Spec(A)$. Let $X \xrightarrow{f} S$ a proper morphism and X_0 the closed fiber. Then the functor*

$$F\tilde{A}\check{l}t(X) \to F\tilde{A}\check{l}t(X_0), \qquad U \mapsto U \times_S X_0$$

Is an equivalence of categories.

Proof. Fully Faithfulness Let $X \to X$ and $X'' \to X$ two finite Ãľtale coverings. Then any X-morphism and any X_0-morphism is determined by the graph

$$\Gamma_\phi : X' \to X' \times_X X''$$

$$\Gamma_{\phi_0} : X'_0 \to X'_0 \times_{X_0} X''_0$$

They are finite Ãľtale and a closed immersions, hence their image is an open and closed subset, and by proposition 2.1.1 we conclude.

Essential Surjectivity Consider $X'_0 \to X_0$ a finite Ãľtale covering, we need to lift it to $X' \to X$ finite Ãľtale covering. We need two lemmas

Theorem. *Let S be a scheme. Let $S_0 \subseteq S$ be a closed subscheme with the same underlying topological space. The functor*

$$X \mapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{schemes X \tilde{A}\check{l}tale\ over\ S\} \leftrightarrow \{schemes X_0 \tilde{A}\check{l}tale\ over S_0\}$$

Proof. [Sta, Tag 039R]

Theorem. *Let $(f, f_0) : (X, X_0) \to (Y, Y_0)$ be a morphism of thickenings. Assume f and f_0 are locally of finite type and $X = Y \times_{Y_0} X_0$. Then f is finite if and only if f_0 is finite*

Proof. [Sta, Tag 09ZW]

We deduce that finite Ãľtale coverings do not depend on nilpotent elements, so X'_0 extends uniquely to a finite Ãľtale covering $X'_k \to X_k$ for all $k \geq 0$. In particular we have a finite Ãľtale covering $\mathfrak{X}' \to \mathfrak{X}$ over the formal completion of X along X_0.
We have Grothendieck's Algebrizarion:

Theorem. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = Spec(A)$ and $S_n = Spec(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd}
X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[r] \arrow[d] & X'_3 \arrow[r] \arrow[d] & \dots \\
X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \dots
\end{tikzcd}$$

of schemes with cartesian squares. Assume that

(1) $X'_n \to X_n$ is a finite morphism, and

(2) $X'_1 \to S_1$ is proper.

Then there exists a finite morphism of schemes $X' \to X$ such that $X'_n = X' \times_S S_n$. Moreover, X' is proper over S.

Proof. [Sta] Lemma 29.25.2

So we deduce that $\mathfrak{X}'$ is the formal completion of a finite Ã l'tale morphism $\bar{X}' \to X \times_A Spec(\hat{A})$
By passage to limit ([Sta], Lemma 31.13.3) we can now restrict to the case when A is the henselization of a $\mathbb{Z}$-algebra of finite type. We have the functor

$$\{A - alg.\} \to Set \quad B \mapsto \{\text{Finite Ã l'tale coverings over } X \times_A Spec(B)\}$$

This functor is locally of finite presentation: if B_i is a filtered inductive system of A-algebras and $B = \varinjlim B_i$, then

$$\{F\tilde{A}l't(X \times_A Spec(\varinjlim B_i))\} = \{F\tilde{A}l't(\varprojlim X \times_A Spec(B_i))\} = \varinjlim \{F\tilde{A}l't(X \times_A Spec(B_i))\}$$

We can apply Artin's Approximation theorem:

Theorem. *Let R be a field or an excellent DVR and let A be the henselization of an R-algebra of finite type at a prime ideal, let $\mathfrak{m}$ be a proper ideal of A and $\hat{A}$ the $\mathfrak{m}$-adic completion of A. Let F be a functor locally of finite presentation, then given any $\bar{\xi} \in F(\hat{A})$ and any integer c, there is a $\xi \in F(A)$ such that*

$$\xi = \bar{\xi} \ (mod \mathfrak{m}^c)$$

i.e. they have the same image via the induced maps over $F(A/\mathfrak{m}^c)$

Proof. [Art69], Theorem 1.12

So in our case, considering $\bar{X}' \in F\tilde{A}l't(X \times_A Spec(\hat{A}))$ as before, there exists $X' \in F\tilde{A}l't(X)$ such that they coincide over X_k.

□

2.2 Reduction to simpler cases

2.2.1 Constructible Sheaves

Definition 2.2.1. An abelian sheaf F on $X_{\tilde{A}l't}$ is *locally constant constructible* (l.c.c) if it is representable represented by a finite Ã l'tale covering of X. Equivalently (see [Fu11, Proposition 5.8.1]), if F is locally constant with finite stalks, and so there exits a finite Ã l'tale morphism $\pi : X' \to X$ such that $\pi^* F$ is constant.

Definition 2.2.2. An abelian sheaf F on $X_{Ãľt}$ is *constructible* if it verifies one of the following equivalent conditions:

(i) There exists a finite surjective family of subschemes X_i such that F_{X_i} is l.c.c.

(ii) There exists a finite family of finite morphisms $X_i' \xrightarrow{p_i} X$ and constant sheaves defined by finite groups C_i over X_i' and a monomorphism

$$F \rightarrowtail \prod_i p_{i*}C_i$$

It is easy to see that constructible sheaves are an abelian category, and moreover if F is constructible and $F \xrightarrow{u} G$ is a morphism of sheave, then $Im(u)$ is constructible.

Lemma 2.2.3. *Every torsion sheaf F is a filtered colimit of constructible sheaves.*

Proof. Let $j : U \to X$ an Ãľtale scheme of finite type, $\xi \in F(U)$ such that $n\xi = 0$. It defines a morphism of sheaves

$$j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U) \to F$$

such that if $\bar{s}$ is a geometric point where U is an Ãľtale neighborhood, then $(j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)_{\bar{s}} = \mathbb{Z}/n\mathbb{Z}$ and the morphism

$$\mathbb{Z}/n\mathbb{Z} \to F_{\bar{s}} \quad m \mapsto m\xi_{\bar{s}}$$

$j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)$ is constructible: $j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)_{j(U)} = \underline{\mathbb{Z}/n\mathbb{Z}}_U$ is represented by $U \coprod U ... \coprod U$ n times, and it is of course a finite Ãľtale covering of U, hence $(\underline{\mathbb{Z}/n\mathbb{Z}}_U)_{j(U)}$ is l.c.c.
On the other hand, since X is noetherian, it's quasi-compact, and $j(U)$ is open since j is Ãľtale. So there is a finte family of open subschemes U_i of X such that $X \setminus j(U) = \cup U_i$, and since open immersions are Ãľtale and $j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)_{|U_i} = 0$ by definition of $j_!$, $j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)_{|U_i}$ are l.c.c. represented by the empty set.
So we have a finite surjective family of subscheme $\{j(U), U_i\}$ such that $j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)$ is locally *l.c.c.*, hence $j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_U)$ is constructible, and in particular, the image is constructible. It is clear that

$$F = \varinjlim_U Im(j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_{|U}))$$

where n and j depend on U □

Definition 2.2.4. Let $\mathcal{C}$ be an abelian category and $T : \mathcal{C} \to \mathcal{A}b$ a functor from $\mathcal{C}$ to the abelian groups. T is *ÃľffaÃğable* if $\forall\ A \in \mathcal{C}$, $\forall\ \alpha \in T(A)$ there is an object $M \in \mathcal{C}$ and a monomorphism $A \rightarrowtail M$ such that $Tu(\alpha) = 0$

Lemma 2.2.5. *The functors*

$$H^p(X_{et}, \_) : \{Constructible\ sheaves\} \to \mathcal{A}b$$

are ÃľffaÃğable $\forall\ p \geq 0$

Proof. It is enough to see that if F is costructible sheaf then it is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some n by definition, so we can embed $F \rightarrowtail G$ into an acyclic sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules (e.g. the *GodÃľment* resolution $\prod_{x \in X} i_{x*}F_{\bar{x}}$, which is flasque). G is a torsion sheaf, so it is a filtered colimit of G_i constructible sheaves, so $F \rightarrowtail G_i$ and since the cohomology commutes with colimits, we have

$$H^p(X_{\text{Ãľt}}, F) \twoheadrightarrow \varinjlim H^p(X_{\text{Ãľt}}, G_i) \quad \alpha \mapsto 0$$

So $\forall\ \alpha \in H^p(X_{\text{Ãľt}}, F)\ \exists\ i$ such that $h : F \rightarrowtail G_i$ is mono and

$$H^p(X_{\text{Ãľt}}, F) \xrightarrow{h^p} H^p(X, G_i) \quad h^p(\alpha) = 0$$

□

We need now a technical lemma on *ÃľffaÃğable* cohomological functors in order to proceed:

Lemma 2.2.6. *Let $\phi^\bullet : T^\bullet \to T'^\bullet$ a morphism of cohomological δ-functors $\mathcal{C} \to \mathcal{A}b$ such that T^q is effaÃğable $\forall\ q$. Let $\mathcal{E}$ be a subclass of $Ob(\mathcal{C})$ such that $\forall\ A \in \mathcal{C}\ \exists\ M \in \mathcal{E}$ and a monomorphism $A \rightarrowtail M$. Then TFAE:*

(i) $\phi^q(A)$ is bijective $\forall\ q \geq 0,\ A \in \mathcal{C}$

(ii) $\phi^0(M)$ is bijective and $\phi^q(M)$ is surjective $\forall\ q > 0,\ M \in \mathcal{E}$

(iii) $\phi^0(A)$ is bijective $\forall\ A \in \mathcal{C}$ and T'^q is ÃľffaÃğable $\forall q > 0$

Proof. by induction

So we can now prove the key proposition:

Proposition 2.2.7. *Let X_0 be a subscheme of X. Suppose that $\forall\ n \geq 0$ and for all $X' \to X$ finite over X the canonical map*

$$H^p(X'_{et}, \mathbb{Z}/n\mathbb{Z}) \to H^p(X'_{0\text{Ãľt}}, \mathbb{Z}/n\mathbb{Z})$$

where $X'_0 = X' \times_X X_0$ is bijective for $q = 0$ and surjective for $q > 0$. Then for all F torsion over X and $\forall\ q \geq 0$ the canonical map

$$H^p(X'_{et}, F) \to H^p(X'_{0\text{Ãľt}}, F)$$

is bijective.

Proof. By passage to limit it is enough to show it for F constructible. Consider, with the notation of *Lemma 9*:

- $\mathcal{C}$ as the category of constructible sheaves,
- $T^\bullet = H^\bullet(X_{\text{Ãľt}})$,
- $T'^\bullet = H^\bullet(X_{0\text{Ãľt}})$,
- $\mathcal{E}$ as the category of constructible sheaf of the form $\prod p_{i*}C_i$ where $p_i : X_i \to X$ is finite and C_i is constant and finite.

□

2.2.2 DÃĽvissages and reduction to the case of curves

Definition 2.2.8. An *elementary fibration* is a morphism of schemes $X \longrightarrow S$ such that it can be prolonged to form a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] & Y \arrow[l, hook', "j"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Such that:

1. j is an open immersion dense and $X = \bar{X} \setminus Y$
2. $\bar{f}$ is projective, smooth with irreducible fibers of dimension 1
3. g is finite ÃĽtale with nonempty fibers

With this, one can split a proper morphism into elementary fibrations: consider $f : X \longrightarrow Y$ a proper morphism, using Chow's Lemma we have

$$\begin{tikzcd} X \arrow[dr, "f"'] & \bar{X} \arrow[l, "\pi"'] \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

with π birational projective, $\bar{f}$ projective. Considering now $\mathbb{P}^n_S \dashrightarrow \mathbb{P}^1_S$ given by the canonical projection

$$[x_0 : ... : x_n] \longrightarrow [x_0 : x_1]$$

This is defined outside the closed subset $Y = Z(x_0, x_1) \cong \mathbb{P}^{n-2}$, so if we consider P the blow-up of $\mathbb{P}^n$ on Y, we get a rational map $\phi : P \longrightarrow \mathbb{P}^1_S$ which extends the projections. The blow-up morphism $P \longrightarrow \mathbb{P}^n$ has fibers of dimensions ≤ 1 and is locally isomorphic to $\mathbb{P}^{n-1}$. In this way one can split a proper morphism into a chain

$$X = X_n \xrightarrow{f_n} X_{n-1} \longrightarrow ...X_1 \xrightarrow{f_1} X_0 = S$$

where all the f_j have fibers of dimension ≤ 1. Hence if $\bar{s}$ is a geometric point of X and assuming that the proper base change theorem holds for relative dimension 1, one has that at every step

$$\begin{tikzcd} X^{(i)}_s \arrow[r] \arrow[d] & X_{\mathcal{O}_{X_i,s}} \arrow[r] & X_{i+1} \arrow[d, "f_i"] \\ s \arrow[r, "\bar{s}"] & Spec(\mathcal{O}_{X_i,s}) \arrow[r] & X_i \end{tikzcd}$$

and since $X^{(i)}_s \longrightarrow Spec(\mathcal{O}_{X_i,s})$ is a proper S-scheme with relative dimension ≤ 1, hence the theorem holds by assumption and rebuilding we have the theorem for $X \longrightarrow S$

2.3 End of the proof

Using the previous reductions, we have reduce ourselves to consider $X \to S$ a proper S-scheme where $S = Spec(A)$ with A a noetherian strictly henselian ring with residue field k, X_0 the closed fiber with dimension ≤ 1 and $n \geq 1$, we need to prove that the canonical morphism

$$H^q(X, \mathbb{Z}/n\mathbb{Z}) \to H^q(X_0, \mathbb{Z}/n\mathbb{Z})$$

is bijective for $q = 0$ and surjective for $q \geq 1$.
The case with $q = 0$ and 1 has already been seen, and one has thet $H^q(X_0, \mathbb{Z}/n\mathbb{Z}) = 0$ for $q \geq 3$ since $X_0 = X \times_S Spec(k)$ is a proper curve over a separably closed field. So we need to prove it for $q = 2$ and WLOG we can suppose $n = \ell^r$ for some prime number ℓ

2.3.1 Proof for $\ell = char k$

We can consider Artin-Schreier exact sequence and we obtain the long exact sequence in cohomology

$$H^1_{Zar}(X_0, \mathcal{O}_{X_0}) \xrightarrow{(F-id)^1} H^1_{Zar}(X_0, \mathcal{O}_{X_0}) \to H^2_{\text{Ãľt}}(X_0, \mathbb{Z}/p\mathbb{Z}) \to H^2_{Zar}(X_0, \mathcal{O}_{X_0})$$

We have that:

Theorem. *Let $\pi : X \to Y$ be a proper morphism of schemes , Y locally Noetherian, $y \in Y$ and $dim(X_y) = d$. Then for any coherent $\mathcal{O}_X$-modules F*

$$(R^q\pi_*F)_y = 0 \quad q > d$$

Proof. [Sta, Tag 02V7]

We have that if $\pi : X_0 \to Spec(k)$ is the canonical map, then $H^2_{Zar}(X_0, \mathcal{O}_{X_0}) = R^2\pi_*(\mathcal{O}_{X_0})_y = 0$, hence

$$H^1_{Zar}(X_0, \mathcal{O}_{X_0}) \xrightarrow{(F-id)^1} H^1_{Zar}(X_0, \mathcal{O}_{X_0}) \to H^2_{\text{Ãľt}}(X_0, \mathbb{Z}/p\mathbb{Z}) \to 0$$

is exact.
Recall that if $\bar{k}$ is the algebraic closure of the separably closed field k, then

$$\bar{k} = \varinjlim k_i$$

where k_i/k are finite purely inseparable extensions. So $X \times_k Spec(k_i) \to X$ is a finite surjective radiciel morphism, so

$$H^i(X, F) \cong H^i(X \times_{Spec(k)} Spec(k_i), F)$$

So we can suppose k algebraically closed.
We need now a technical lemma:

Theorem. *Let k be an algebraically closed field of characteristic p and V a finite-dimensional k-vector space, $F : V \to V$ a Frobenius map, i.e. $F(\lambda v) = \lambda^p F(v)\ \forall\ \lambda \in k,\ v \in V$. Then $F - id : V \to V$ is surjective*

Proof. [Sta, Tag 0DV6]

Then using Grothendieck's Coherency Theorem

Theorem. *Let $f : X \to Y$ proper, Y noetherian, F a coherent $\mathcal{O}_X$-module. Then $R^q f_* F$ is a coherent $\mathcal{O}_Y$-module.*
In particular if $Y = Spec(A)$, then since $R^q f_* F = H^q(X, F)^{\sim}$, so $H^q(X, F)$ is a finite A-module.

Proof. [MO15, 7.7.2]

Combining this two results again on $\pi : X_0 \to Spec(k)$, one has that $(F - id)^1$ is surjective, so $H^2(X_0, \mathbb{Z}/p\mathbb{Z}) = 0$

2.3.2 Proof for $\ell \neq char(k)$

Using Kummer exact sequence one has the following commuative diagram

$$\begin{array}{ccc} Pic(X) & \xrightarrow{\alpha} & H^2(X, \mathbb{Z}/\ell^r\mathbb{Z}) \\ \downarrow & & \downarrow \\ Pic(X_0) & \xrightarrow{\beta} & H^2(X_0, \mathbb{Z}/\ell^r\mathbb{Z}) \end{array}$$

It can be shown that for all proper curves over a separably closed field the map β is surjective ([AGV72, IX.4.7])
So it is enough to show that

Proposition 2.3.1. *Let S be the spectrum of an henselian ring, X and S-scheme and X_0 the closed fiber. Then the canonical map $Pic(X) \to Pic(X_0)$ is surjective*

Proof. Since X_0 is a curve, it is enough to prove that the canonical map $Div(X) \to Div(X_0)$ is surjective.
Every divisor on X_0 is a linear combination of divisor with support in closed points. So consider x a closed point of X_0, $t_0 \in \mathcal{O}_{X_0,x}$ a regular non invertible element and D_0 the divisor of local equation t_0. Consider an open neighborhood $U \subseteq X$ of x and let $t \in \mathcal{O}_U(U)$ a lifting of t_0. Then consider Y the close subset of U defined by $t = 0$. Taking U small enough, one can suppose that $Y \cap X_0 = \{x\}$. Then Y is quasi-fintie in x over S.
Then by the characterizations of henselian local rings ([Sta, Tag 04GG]) $Y = Y_1 \coprod Y_2$ with Y_1 finite and $Y_2 \cap X_0 = \emptyset$. And since X is separated over S, Y_1 is closed in X since finite $\Rightarrow$ proper.
So by choosing U small enough, one can suppose $Y = Y_1$, hence Y is closed in X. So one can define a divisor D on X corresponding to Y and one have $D_{|X \setminus Y} = 0$ and $D_{|U} = div(t)$. Then $D_{|X_0} = D_0$. □

2.4 Proper support

2.4.1 Extension by zero

Definition 2.4.1. Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ be a LEX additive functor between abelian categories. We can define the **mapping cylinder** of f as the following category $\mathcal{C}$:

- $Ob(\mathcal{C})$ are triplets (A, B, ϕ) such that $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\phi \in \mathrm{Hom}_{\mathcal{B}}(B, fA)$
- $\xi : (A, B, \phi) \to (A', B', \phi')$ is given by $\xi_A \in \mathrm{Hom}_{\mathcal{A}}(A, A')$, $\xi_B \in \mathrm{Hom}_{\mathcal{B}}(B, B')$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\xi_B"] & fA \arrow[d, "f\xi_A"] \\ B' \arrow[r, "\phi'"] & fA' \end{tikzcd}$$

It is immediate to see that $\mathcal{C}$ is abelian and

$$(A', B', \phi') \to (A, B, \phi) \to (A'', B'', \phi'')$$

is exact if and only if

$$A' \to A \to A'' \qquad B' \to B \to B''$$

are exact.

Definition 2.4.2. We can define functors:

$$\begin{array}{llll} j^* : \mathcal{C} \to \mathcal{A} & (A, B, \phi) \mapsto A & i^* : \mathcal{C} \to \mathcal{B} & (A, B, \phi) \mapsto B \\ j_* : \mathcal{A} \to \mathcal{C} & A \mapsto (A, fA, id) & i_* : \mathcal{B} \to \mathcal{C} & B \mapsto (0, B, 0) \\ j_! : \mathcal{A} \to \mathcal{C} & A \mapsto (A, 0, 0) & i^! : \mathcal{C} \to \mathcal{B} & (A, B, \phi) \mapsto ker(\phi) \end{array}$$

(i) It is trivial that $j_! \dashv j^* \dashv j_*$ and $i^* \dashv i_* \dashv i^!$, just check on the Hom. In particular j_* and $i^!$ are left exact.

(ii) By definition, $j^*, j_!, i^*$ and i_* are exact.

(iii) By definition, j_* and i_* are fully faithful.

(iv) By definition, $i^* j_* = f$ and $i^* j_! = i^! j_! = i^! j_* = j^* i_* = 0$

We need now a technical lemma:

Theorem. *Consider abelian categories $\mathcal{A}$, $\mathcal{C}'$ and $\mathcal{B}$ and functors*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "j_*"] & \mathcal{C}' \arrow[l, shift left, "j^*"] \arrow[r, shift left, "i^*"] & \mathcal{B} \arrow[l, shift left, "i_*"] \end{tikzcd}$$

such that:

a) $j^* \dashv j_*$ *and* $i^* \dashv i_*$

b) j^* *and* i^* *are exact*

c) j_* *and* i_* *are fully faithful*

d) For $C \in \mathcal{C}'$ *we have* $j^*C = 0$ *if and only if* $C = i_*B$ *for some* $B \in \mathcal{B}$

Then the functor $f = i^*j_*$ *is left exact and additive, and the functor*

$$C \mapsto (j^*C, i^*C, i^*C \xrightarrow{i^*\epsilon^j_C} i^*j_*j^*C = fj^*C)$$

is an equivalence between $\mathcal{C}'$ *and the mapping cylinder* $\mathcal{C}$ *of* f

Proof. see [Tam12, 8.1.6]

Consider now the following situation: let X be a scheme and Y a closed subscheme, $U = X \setminus Y$ with the natural structure of open subscheme, let $i : Y \to X$ and $j : U \to X$ the canonical immersions. We have

$$\begin{tikzcd} U_{\text{ÃĽ}t} \arrow[r, shift left, "j_*"] & X_{\text{ÃĽ}t} \arrow[l, shift left, "j^*"] \arrow[r, shift left, "i^*"] & Y_{\text{ÃĽ}t} \arrow[l, shift left, "i_*"] \end{tikzcd}$$

a) and b) are verified.
It can be easily shown ([Tam12, 8.1.1]) that if $f : Y \to X$ is an immersion (i.e. a closed immersion followed by an open immersion), then $\epsilon : f^*f_* \to Id$ is a natural isomorphism, hence f_* is fully faithful. So c) is verified.
It can also be easily shown ([Tam12, 8.1.2]) that i_* induces an equivalence between $Y_{\text{ÃĽ}t}$ and the sheaf of $X_{\text{ÃĽ}t}$ vanishing outside Y, so since by definition $j^*F = 0$ if and only if F vanishes outside Y, d) is verified. So we have proved that

Theorem 2.4.3. *In the situation above, we have an equivalence of categories between* $X_{\text{ÃĽ}t}$ *and the mapping cylinder of* i^*j_* *given by*

$$F \mapsto (j^*F, i^*F, i^*F \xrightarrow{i^*\epsilon^j_F} i^*j_*j^*F)$$

So by the previous construction we have an exact functor $j_! : U_{\text{ÃĽ}t} \to X_{\text{ÃĽ}t}$ and a left exact exact functor $i^! : X_{\text{ÃĽ}t} \to Y_{\text{ÃĽ}t}$. In particular, since $j^*j_! = id$ and $i^*j_* = 0$, we have that if $\bar{x}$ is a geometric point on X and $F \in U_{\text{ÃĽ}t}$

$$(j_!F)_{\bar{x}} = \begin{cases} F_{\bar{x}} & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

$j_!$ is called *extension by zero*

2.4.2 Cohomology with proper support

Considering X a k-scheme of finite type with k algebraically closed, then one has a (not unique) Nagata compactifiation

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr] & \bar{X} \arrow[d] \\ & S \end{tikzcd}$$

where $\overline{X}$ is a proper k-scheme of finite type and j is an open immersion. Then one can define for any torsion sheaf F

$$H^i_c(X, F) := H^i(\overline{X}, j_! F)$$

This is independent from the choice of the compactifiation: if $\overline{X}_1$ and $\overline{X}_2$ are two compactifiations, then $\overline{X}_1 \times_X \overline{X}_2$ is again a compactifiation and $\overline{X}_1 \times_X \overline{X}_2 \to X_t$ is proper for $t = 0, 1$, so we have to check only the situation where we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_2"] \arrow[dr, "j_1"'] & \bar{X}_2 \arrow[d, "p"] \\ & \bar{X}_1 \end{tikzcd}$$

with p proper.

Lemma 2.4.4. *$p_* j_{2!} F = j_{1!} F$ and $R^q p_* j_{2!} F = 0$ for $q > 0$*

Proof. Equality holds if and only if it holds for every geometric point $\bar{s}$, so one uses the proper base change for p

$$(p_* j_{2!} F)_{\bar{s}} \cong H^0((\overline{X}_2)_s, j_{2!} F) \cong \begin{cases} F_{\bar{s}} & \text{if } s \in \overline{X}_1 \\ 0 & \text{otherwise} \end{cases}$$

and $H^i((X_2)_{\bar{s}}, j_{2!} F) = 0$ for $i > 0$ since j_2 is an open immersion and the fibers are of dimension ≤ 1 □

This lemma says that in the derived categories we have $Rp_* j_{2!} = j_{1!}$ and $Rp_* j_{2!} F = p_* j_{2!} F$, so we conclude. In particular one can define for any separated morphism of finite type between Noetherian schemes $X \xrightarrow{f} Y$ a higher direct image with proper support considering $\bar{X} \xrightarrow{\bar{f}} Y$ proper and $j : X \to \bar{X}$ open, hence for any torsion sheaf F

$$R^p f_! F := R^p f_* j_! F$$

This follows directly from the proper base change:

Theorem 2.4.5. *Let $X \xrightarrow{f} Y$ be a separated morphism of finite type of Noetherian schemes. Let $Y' \xrightarrow{g} Y$ be a morphism of schemes. Set $X' = X \times_Y Y'$ and consider the cartesian* *diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then for any F torsion sheaf on X_{et} one has an isomorphism of Y'-sheaves

$$g^* R^p f_! F \cong R^p f'_! g'^* F$$

Proof. Just apply the proper base change on $\bar{X}$ and $j_! F$

Remark 2.4.6. One can define the direct image with compact support even for non torsion sheaves, but in this case we will need to fix a compactifiation. In the next chapter, this will be done considering a totally imaginary number field K, then $Spec(\mathcal{O}_K) \to Spec(\mathbb{Z})$ is finite, hence proper. So consider $f : U \hookrightarrow Spec(\mathcal{O}_K)$ an open, for every sheaf F on U we define

$$H^r_c(U, F) = H^r(X, j_! F)$$

This will allow us to generalize results given on torsion sheaves.

2.4.3 Applications

Theorem 2.4.7. *Let $f : X \to S$ be a separated morphism of finite type of relative dimension $\le n$[1] and F a torsion sheaf on X. Then $R^q f_! F = 0$ for all $q > 2n$*

Proof. Take $\bar{y}$ a geometric point of Y. Then for the previous theorem we have:

$$(R^p f_! F)_{\bar{y}} = H^q_c(X_y, F_{X_y})$$

Hence it is enough to prove that for $X \to Spec(k)$ separated of finite type of dimension n with k separably closed, $H^q_c(X, F) = 0$ for $q > 2n$. Use induction on n: if $dim(n) = 0$, it's true since $\Gamma(X, \_)$ is exact. So suppose $dim(X) \ge 1$. We have that $dim(X \setminus X_{red}) = 0$, so it is enough to prove it for X reduced. If an irrdeducible component D has dimension $< n$, then for the exact sequence

$$0 \to F_D \to F \to F_{X \setminus D} \to 0$$

we can suppose X of pure dimension n. Take $\eta_1 \cdots \eta_m$ be the generic points and take $t_i \in k(\eta_i)$ transcendent over k. Take U_i an disjoint open affine irreducible neighborhood of η_i such that $t_i \in \mathcal{O}_X(U_i)$, hence we have a morphism $T \mapsto t_i$ which gives a morphism

$$U_i \to \mathbb{A}^1_k$$

Since the set $\{x \in X : \mathcal{O}_{X,x}$ is flat over $k[T]\}$ is open ([Fu11, 1.5.8]), we can consider U_i flat over $\mathbb{A}^1_k$. Hence we have for all closed points $y \in \mathbb{A}^1_k$ and $x \in U_i$ who lies above y

$$dim_{Kr}(\mathcal{O}_{U_i,x} \otimes_{\mathcal{O}_{\mathbb{A}^1_k,y}} k(y)) = dim_{Kr}(\mathcal{O}_{U_i,x}) - dim_{Kr}(\mathcal{O}_{\mathbb{A}^1_k,y}) \le n - 1$$

[1] i.e. for all geometric points s of S $dim(X_s) \le n$

so the fibers above closed points have dimension $\leq n-1$. So if $U = \cap U_i \xrightarrow{f} \mathbb{A}^1_k$, we have by induction hypothesis

$$R^p f_! F_{|U} = 0 \text{ if } q > 2(n-1)$$

We have by [AGV72, IX,5] that for all torsion sheaves G over $\mathbb{A}^1_k$ we have

$$H^p_c(\mathbb{A}^1_k, G) = H^p(\mathbb{P}^1_k, [1:\_]_! G) = 0 \text{ if } p > 2$$

So by the spectral sequence

$$H^p(\mathbb{A}^1, R^q f_! F) \Rightarrow H^{p+q}_c(U, F)$$

we get that $H^q_c(U, F) = 0$ for $q > 2n$.
By construction, $dim(X \setminus U) \leq n-1$ since U contains all the generic points. So again by induction hypothesis

$$H^q_c(X \setminus U, F_{|X \setminus U}) = 0 \text{ if } q > 2(n-1)$$

Hence we conclude by the long exact sequence

$$\to H^i_c(U, F) \to H^i_c(X, F) \to H^i(X \setminus U, F) \to \dots$$

□

Theorem 2.4.8. *Let $f : X \to S$ be a separated morphism f finite type and F a constructible sheaf on X. Then $R^q f_! F = 0$ is constructible*

Proof. [Del, Arcata IV 6.2] □

Theorem 2.4.9 (Projection formula)**.** *Let $f : X \to Y$ be a compactifiable morphism, let Λ be a torsion ring. Then for any $K \in D^-(X, \Lambda)$ and $L \in D^(Y, \Lambda)$ we have a canonical isomorphism*

$$L \otimes^{\mathbb{L}}_\Lambda \mathcal{R} f_! K \cong \mathcal{R} f_!(f^* L \otimes^{\mathbb{L}}_\Lambda K)$$

Proof. Let $X \xrightarrow{j} \overline{X} \xrightarrow{\bar{f}} Y$ the compactification, so $\mathcal{R} f_! = \mathcal{R}\bar{f}_* j_!$. Let $M^\bullet$ a complex of sheaves of Λ-modules on X and $N^\bullet$ a complex of sheaves of Λ-modules on $\overline{X}$. Then using the mapping cylinder we have $N = (j^*N, i^*N, \phi)$ and

$$j_!(j^*N \otimes_\Lambda M) \cong (j^*N \otimes_\Lambda M, 0, 0) \cong N \otimes j_! M$$

So if $M^\bullet$ is a complex of flat modules quasi isomorphic to $K^\bullet$ and $N^\bullet$ is quasi isomorphic to $\bar{f}^* L^\bullet$, the isomorphism gives a quasi isomorphism

$$j_!(j^* \bar{f}^* L \otimes^{\mathbb{L}}_\Lambda K) \cong \bar{f}^* L \otimes^{\mathbb{L}} j_! K$$

Hence, since

$$\mathcal{R} f_!(f^* L \otimes^{\mathbb{L}}_\Lambda K) = \mathcal{R}\bar{f}_* j_!(j^* \bar{f}^* L \otimes^{\mathbb{L}}_\Lambda K) = \mathcal{R}\bar{f}_*(\bar{f}^* L \otimes^{\mathbb{L}}_\Lambda j_! K)$$

it is enough to prove that the canonical morphism induced by the adjunction $L \to \bar{f}_* \bar{f}^* L$

$$L \otimes^{\mathbb{L}}_\Lambda \mathcal{R}\bar{f}_* j_! K \to \mathcal{R}\bar{f}_*(\bar{f}^* L \otimes^{\mathbb{L}}_\Lambda j_! K)$$

is an isomorphism. Let $s \to Y$ be a geometric point of Y and $\bar{f}_s : \overline{X}_s \to s$ be the base change. For the proper base change we get

$$\begin{aligned}(L \otimes^{\mathbb{L}}_{\Lambda} \mathcal{R}\bar{f}_* j_! K)_s &\cong L_s \otimes^{\mathbb{L}}_{\Lambda} \mathcal{R}\bar{f}_{s*}(j_! K)_{\overline{X}_s} \\ (\mathcal{R}\bar{f}_*(\bar{f}^* L \otimes^{\mathbb{L}}_{\Lambda} j_! K))_s &\cong \mathcal{R}\bar{f}_{s*}(\bar{f}^* L \otimes^{\mathbb{L}}_{\Lambda} j_! K)_{\overline{X}_s}\end{aligned}$$

□

Chapter 3

Geometry: PoincarÃľ duality

3.1 Trace maps

Fix a base scheme S and n invertible on S. Then we define for all d and any sheaf F of $\mathbb{Z}/n\mathbb{Z}$-modules:

$$\mathbb{Z}/n\mathbb{Z}(d) = \begin{cases} \mu_n^{\otimes d} & \text{if } d > 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } d = 0 \\ \mathcal{H}om(\mu_n^{\otimes -d}, \mathbb{Z}/n\mathbb{Z}) & \text{if } d < 0 \end{cases} \qquad F(d) = \mathbb{Z}/n\mathbb{Z}(d) \otimes F$$

Let $f : X \to S$ be a smooth S-compactifiable morphism of relative dimension d. The aim of this section is to construct a canonical morphism

$$Tr_{X/Y} : R^{2d} f_! f^* F(d) \to F$$

If f is Ãľtale, then $d = 0$ and $f_!$ is left adjoint to f^*, hence we define Tr_f as the counit $Tr_f : f^* f_! F \to F$. Let X be an integral proper smooth curve over an algebraically closed field k. By theorem B.9.4 we have

$$H^2(X, \mu_n) \cong Pic(X)_n \xrightarrow[\sim]{deg} \mathbb{Z}/n\mathbb{Z}$$

Hence we get $Tr_{X/k} : H^2(X, \mathbb{Z}/n\mathbb{Z}(1)) \to \mathbb{Z}/n\mathbb{Z}$ this morphism.
If X is smooth irreducible over k algebraically closed, then if $\overline{X}$ is its compactification, $\overline{X} \setminus X$ has dimension 0, hence it is a finite set of points, so $H^1(\overline{X} \setminus X, \mu_n) = H^2(\overline{X} \setminus X, \mu_n) = 0$, so for the long exact sequence we get

$$H^1_c(X, \mathbb{Z}/n\mathbb{Z}(1)) \cong H^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}(1))$$

and we define $Tr_{X/k}$ to be the composition of this isomorphism and $Tr_{\overline{X}/k}$.
If X is smooth over k algebraically closed, then the irreducible components $X_1 ... X_m$ are the connected components, hence

$$H^2_c(X, \mathbb{Z}/n\mathbb{Z}(1)) \cong \oplus H^2_c(X_i, \mathbb{Z}/n\mathbb{Z}(1))$$

and we define $Tr_{X/k} := \oplus Tr_{X_i/k}$.
Consider now, $f : X \to Y$ Ã©tale and Y/k a smooth curve over k alg closed. Since $f^*\mu_{n,Y} = \mu_{n,X}$, the counit gives a morphism $\mu_{n,Y} \to f_!\mu_{n,X}$

$$S_{X/Y} : H^2_c(X, \mu_{n,X}) \cong H^2_c(Y, f_!\mu_{n,X}) \to H^2_c(Y, \mu_{n,Y})$$

Lemma 3.1.1. *In the situation above, we have* $Tr_{X/k} = Tr_{Y/k}S_{X/Y}$

Sketch of proof. (see [Fu11, 8.2.1]) Consider the morphism on the compactification $\bar{f}$ definde by

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\bar{f}} & \overline{Y} \\ {\scriptstyle j^X}\uparrow & & {\scriptstyle j^Y}\uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

It is finite and flat, so $\bar{f}_*\mathcal{O}_{\overline{X}}$ is locally free of finite type over $\mathcal{O}_{\overline{Y}}$. Consider V such that $(\bar{f}_*\mathcal{O}_{\overline{X}})_{|V}$ is free, then for all $s \in \bar{f}_*\mathcal{O}_{\overline{X}}(V)$ we have an endomorphism induced by the multiplication by s, hence we have a morphism

$$s \mapsto det(s) : \bar{f}_*\mathcal{O}_{\overline{X}}(V) \to \mathcal{O}_{\overline{Y}}(V)$$

Hence we have a morphism of sheaves

$$det : \bar{f}_*\mathcal{O}^\times_{\overline{X}} \to \mathcal{O}^\times_{\overline{Y}}$$

which induces a morphism of Ã©tale sheaves

$$det : \bar{f}_*\mathbb{G}_{m\overline{X}} \to \mathbb{G}_{m\overline{Y}}$$

And since $\bar{f}_*$ is finite, it is exact, so we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_{n,Y} & \longrightarrow & \bar{f}_*\mathbb{G}_{m\overline{X}} & \xrightarrow{n} & \bar{f}_*\mathbb{G}_{m\overline{X}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle det} & & \downarrow{\scriptstyle det} & & \\ 0 & \longrightarrow & \mu_{n,Y} & \longrightarrow & \mathbb{G}_{m\overline{Y}} & \xrightarrow{n} & \mathbb{G}_{m\overline{Y}} & \longrightarrow & 0 \end{array}$$

We define $Tr_{\overline{X}/\overline{Y}}$ to be the map on the kernel. In fact, we have that if $\bar{y}$ is a geometric point of Y, then

$$(f_*\mu_n)_{\bar{y}} = \bigoplus_{x \in X_y} \Gamma(Spec(\mathcal{O}^{sh}_{\overline{X},\bar{x}}), \mu_n)$$

So for any $(\lambda_x) \in (f_*\mu_n)_{\overline{y}}$

$$Tr_{\overline{X}/\overline{Y}}((\lambda_x)) = \prod \lambda_x^{n_x}$$

with $n_x = rank_{\mathcal{O}^{sh}_{\overline{Y},\bar{y}}}(\mathcal{O}^{sh}_{\overline{X},\bar{x}})$ Then we have a commutative diagram

$$\begin{array}{ccccc} j^Y_! f_!\mu_n & \xrightarrow{\sim} & \bar{f}_* j^Y_! \mu_n & \xrightarrow{\bar{f}_*(Tr_{\overline{X}/X})} & \bar{f}_*\mu_n \\ & \searrow & \downarrow{\scriptstyle j^Y_! Tr_{X/Y}} & & \downarrow{\scriptstyle Tr_{\overline{X}/\overline{Y}}} \\ & & j^Y_!\mu_n & \xrightarrow{Tr_{\overline{Y}/Y}} & \mu_n \end{array}$$

Which gives a commutative diagram

$$\begin{tikzcd} H^2_c(X, \mu_n) \arrow[r, "\sim"] \arrow[d, "S_{X/Y}"] & H^2(\overline{X}, \mu_n) \arrow[d, "S_{\overline{X}/\overline{Y}}"] \\ H^2_c(Y, \mu_n) \arrow[r, "\sim"] & H^2(\overline{Y}, \mu_n) \end{tikzcd}$$

Hence it is enough to prove that $Tr_{\overline{X}/k} = Tr_{\overline{Y}/k} S_{\overline{X}/\overline{Y}}$, but by definition and Kummer theory

$$\begin{tikzcd} Pic(\overline{X}) \arrow[r] \arrow[d, "det"] & H^2(\overline{X}, \mu_n) \arrow[r] \arrow[d, "S_{\overline{X}/\overline{Y}}"] & 0 \\ Pic(\overline{Y}) \arrow[r] & H^2(\overline{Y}, \mu_n) \arrow[r] & 0 \end{tikzcd}$$

So it is enough to prove that the following diagram commutes

$$\begin{tikzcd} Pic(\overline{X}) \arrow[dr, "deg"] \arrow[dd, "det"'] & \\ & \mathbb{Z} \\ Pic(\overline{Y}) \arrow[ur, "deg"] & \end{tikzcd}$$

And this follows form the fact that if $\mathcal{L} \in Pic(\overline{X})$, then as a Cartier divisor it is $\mathcal{L} = (s_i, f^{-1}V_i)$ with $\{V_i\}$ and open cover of Y. Then $det(\mathcal{L}) = (det(s_i), V_i)$, and the assertion follows by the fact that for any closed point y and any $s \in K(X)^\times$, we have

$$v_y(det(s)) = \sum_{x \in X_y} v_x(s)$$

and the theorem comes from the base change $Spec(\hat{\mathcal{O}}_{Y,y}) \to Y$ and for the fact that

$$v_y(det(s)) = v_y(N_{K^{sh}(X)/\{K^{sh}(Y)}(s)) = [k(x) : k(y)]v_x(s) = v_x(s)$$

and since $\mathcal{O}^{sh}_{Y,y}$ is strictly henselian $[k(x) : k(y)] = 1$.

Definition 3.1.2. Let now X be any scheme. For a line bundle $\mathcal{L} \in Pic(X)$, denote $c_1(\mathcal{L})$ its image through the map given by Kummer

$$Pic(X) \twoheadrightarrow H^2(X, \mu_n)$$

And for any $f : X \to Y$ denote $c_{1_{X/Y}}(\mathcal{L})$ the image under the canonical morphism given by the spectral sequence

$$H^2(X, \mu_n) \to \Gamma(Y, R^2 f_* \mu_n)$$

which induces by adjunction a unique morphism

$$H^2(X, \mu_n)_Y \to R^2 f_* \mu_n$$

Where $H^2(X, \mu_n)_Y$ is the constant sheaf associated to the abelian group $H^2(X, \mu_n)$

Proposition 3.1.3. *Let $f : \mathbb{P}^1_Y \to Y$ be the projection, Y any scheme, then the morphism defined by*

$$1 \mapsto c_{1_{X/Y}}(\mathcal{O}_{\mathbb{P}^1_Y}(1)) : \mathbb{Z}/n\mathbb{Z} \to R^2 f_* \mu_n$$

Proof. By proper base change, it is enough to prove it for $Y = Spec(k)$ with k algebraically closed. Hence here we have the isomorphism

$$\bar{c}_1 : Pic(\mathbb{P}^1_k)/nPic(\mathbb{P}^1_k) \cong H^2(\mathbb{P}^1_k, \mu_n)$$

And deg gives the isomorphism

$$Pic(\mathbb{P}^1_k)/nPic(\mathbb{P}^1_k) \to \mathbb{Z}/n\mathbb{Z}$$

So the lemma comes from the fact that $deg(\mathcal{O}_{\mathbb{P}^1_k} = 1$ (it is the hyperplane bundle) □

Define $Tr_{\mathbb{P}^1_Y/Y}$ an the inverse of this isomorphism. Consider now $f : \mathbb{A}^1_Y \to Y$ the projection and $j : \mathbb{A}^1_Y \to \mathbb{P}^1_k$ the compactification, then we have $Tr_{\mathbb{A}^1/Y} : R^2 f_! \mu_n \to \mu_n$ as the composition of

$$R^2 f_! \mu_{n,\mathbb{A}^1_k} = R^2 \bar{f}_* j_! \mu_{n,\mathbb{A}^1_k} \xrightarrow{R^2 \bar{f}_* Tr_{\mathbb{A}^1_k/\mathbb{P}^1_k}} R^2 \bar{f}_* \mu_{n,\mathbb{P}^1_k} \xrightarrow{Tr_{\mathbb{P}^1_Y/Y}} \mathbb{Z}/n\mathbb{Z}$$

Consider now $g : X \to Y$ and $h : Y \to Z$ smooth compactifiable morphisms of relative dimension d and e respectively. Then $f = hg$ is smooth compactificable of relative dimension $d + e$. Suppose that we have defined:

$$Tr_g : R^{2d} g_! \mathbb{Z}/n\mathbb{Z}(d) \to \mathbb{Z}/n\mathbb{Z} \qquad Tr_h : R^{2e} h_! \mathbb{Z}/n\mathbb{Z}(e) \to \mathbb{Z}/n\mathbb{Z}$$

So since for proper base change $f_!$, $g_!$ and $h_!$ have finite cohomological dimension over torsion sheaves, they define in the derived category:

$$Tr_g : Rg_! \mathbb{Z}/n\mathbb{Z}(d)[2d] \to \mathbb{Z}/n\mathbb{Z} Tr_h : Rh_! \mathbb{Z}/n\mathbb{Z}(e)[2e] \to \mathbb{Z}/n\mathbb{Z}$$

And since $\mathbb{Z}/n\mathbb{Z}(d + e) = \mathbb{Z}/n\mathbb{Z}(d) \otimes^L_{\mathbb{Z}/n\mathbb{Z}} g_* \mathbb{Z}/n\mathbb{Z}(e)$, by the projection formula proposition C.7.8:

$$Rg_!(\mathbb{Z}/n\mathbb{Z}(d) \otimes^L_{\mathbb{Z}/n\mathbb{Z}} g^* \mathbb{Z}/n\mathbb{Z}(e)) \cong R\bar{g}_*(j_! \mathbb{Z}/n\mathbb{Z}(d) \otimes^L_{\mathbb{Z}/n\mathbb{Z}} j_! j^* \bar{g}^* \mathbb{Z}/n\mathbb{Z}(e)) = Rg_! \mathbb{Z}/n\mathbb{Z}(d) \otimes \mathbb{Z}/n\mathbb{Z}(e))$$

So we can consider

$$Rf_!(\mathbb{Z}/n\mathbb{Z}(d + e)[2(d + e)]) \cong Rh_!(Rg_! \mathbb{Z}/n\mathbb{Z}(d)[2d] \otimes^L_{\mathbb{Z}/n\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}(e)[2e]) \xrightarrow{Tr_g}$$
$$Rh_!(\mathbb{Z}/n\mathbb{Z} \otimes^L_{\mathbb{Z}/n\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}(e)[2e]) \xrightarrow{Tr_h} \mathbb{Z}/n\mathbb{Z}$$

Hence this define a map

$$Tr_f : R^{2(d+e)} f_!(\mathbb{Z}/n\mathbb{Z}(d + e)) \to \mathbb{Z}/n\mathbb{Z}$$

And we denote this way of composing traces as

$$Tr_{X/Z} = Tr_{Y/Z} \diamond Tr_{X/Y}$$

So since $\mathbb{A}^n_Y := \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y = \mathbb{A}^1_{\mathbb{A}^{n-1}_Y}$, we can define $Tr_{\mathbb{A}^n_Y/Y}$ considering the dÃľvissage:

$$\mathbb{A}^n_Y \longrightarrow \mathbb{A}^{n-1}_Y \cdots \mathbb{A}^1_Y \longrightarrow Y$$

where the maps are induced by the inclusions

$$\mathbb{Z}[t_1 \cdots t_{d-1}] \longrightarrow \mathbb{Z}[t_1 \cdots t_d]$$

and since $Tr_{\mathbb{A}^1_Y/Y}$ has already been constructed, by composition

$$Tr_{\mathbb{A}^n_Y/Y} = Tr_{\mathbb{A}^1_{\mathbb{A}^{n-1}}/\mathbb{A}^{n-1}_Y} \diamond \cdots \diamond Tr_{\mathbb{A}^1_Y/Y}$$

So if f factorizes as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\tilde{f}"'] & & Y \\ & \mathbb{A}^n_Y \arrow[ur] & \end{tikzcd}$$

with $\tilde{f}$ Ãľtale, we have defined Tr_f, and it can be shown ([Fu11, Lemma 8.2.3]) that it is independent from the factorization.
Finall, if $f : X \longrightarrow Y$ is smooth of relative dimension d, then there is an open cover U_a of X such that

$$\begin{tikzcd} U_a \arrow[rr, "f_{|U_a}"] \arrow[dr, "\tilde{f}_a"'] & & Y \\ & \mathbb{A}^n_{U_a} \arrow[ur] & \end{tikzcd}$$

and there is the exact sequence of sheaves

$$0 \longrightarrow \mathrm{Hom}(R^{2d}(f)_!F, \mathbb{Z}/n\mathbb{Z}) \longrightarrow\longrightarrow \prod \mathrm{Hom}(R^{2d}(f_{U_a})_!F, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \prod \mathrm{Hom}(R^{2d}(f_{U_a \cap U_b})_!F, \mathbb{Z}/n\mathbb{Z})$$

So Tr_f is well defined for all smooth f using the glueing property for sheaves.

3.2 PoincarÃľ duality of curves

3.2.1 Algebraically closed fields

Definition 3.2.1. Let Λ be a ring, X a compactifiable S-scheme of Krull dimension N, $\overline{X}$ is its compactification and $j : X \longrightarrow \overline{X}$ is the open immersion. Then we have the exact functor $j_! : Sh(X, \Lambda) \longrightarrow Sh(\overline{X}, \Lambda)$, since for all F we have by definition $H^i_c(X, F) = \mathrm{Ext}^i_{\overline{X}}(A_{\overline{X}}, j_!F)$. So for all $F, G \in Sh(X, \Lambda)$, we can define as in definition C.5.8 a cup product pairing

$$H^i_c(X, F) \times \mathrm{Ext}^{2N-i}_X(F, G) \longrightarrow H^{2N}_c(X, G)$$

Lemma 3.2.2. *If X is a Noetherian scheme, Λ a ring such that Λ is injective as a Λ-module, F a locally constant sheaf, then $\mathcal{E}xt^q(F, \Lambda) = 0$ for $q > 0$*

Proof. Since being zero is a local property, we may assume F constant associated to a Λ-module M. Consider a free resolution $L_\bullet \to M$ and denote also L^i the constant sheaf associated. Then for all N and all geometric points $\bar{x}$

$$(\mathcal{H}om(L_i, N))_{\bar{x}} = \varinjlim_{x \in U} \mathrm{Hom}_{Sh(X,\Lambda)}(L_i, N_U) = \varinjlim_{x \in U} \mathrm{Hom}_{\Lambda)}(L_i, N(U)) = \mathrm{Hom}_\Lambda(L_i, N_x)$$

Hence since if $N \to I^\bullet$ is an injective resolution of sheaves, $N_x \to I_x$ is an injective resolution of Λ-modules, so

$$(R\mathcal{H}om(L_i, N))_x = \mathcal{H}om(L_i, I^\bullet)_x \cong \mathrm{Hom}_\Lambda(L_i, I_x^\bullet) = R\mathrm{Hom}_\Lambda(L_i, N)$$

But L_i is free, so $R\mathrm{Hom}_\Lambda(L_i, N_x) = \mathrm{Hom}_\Lambda(L_i, N_x)$, hence $\mathcal{E}xt^q(L_i, N) = 0$, so $R\mathcal{H}om(F, N) = \mathcal{H}om(L_\bullet, N)$.
Hence, in our case $R\mathcal{H}om(F, \Lambda) = \mathcal{H}om(L^\bullet, \Lambda)$, and since they are both constant $R\mathcal{H}om(L^\bullet, \Lambda) = R\mathrm{Hom}_\Lambda(L^\bullet, \Lambda) = \mathrm{Hom}_\Lambda(L^\bullet, \Lambda)$ since Λ is Λ-injective by hypothesis, so $\mathcal{E}xt^q(F, \Lambda) = 0$ for $q > 0$ □

Remark 3.2.3. Let X be a smooth curve over an algebraically closed field k, if F is locally constant then $\_ \otimes^{\mathbb{L}} F \dashv R\mathcal{H}om(F, \_)$, so we have that if F is lcc and G is injective:

$$\mathrm{Ext}^q(\Lambda, \mathcal{H}om(F, G)) \cong \mathrm{Ext}^q(F, G)$$

hence if Λ is injective as Λ-module (e.g. $\Lambda = \mathbb{Z}/n\mathbb{Z}$) and $G = \mu_n$, composing with the trace we have

$$H_c^i(X, F) \times H^{2N-i}(X, \mathcal{H}om(F, \mu_n)) \to H_c^{2N}(X, G) \xrightarrow{Tr} \mathbb{Z}/n\mathbb{Z}$$

So the aim of the section is to prove that if $N = 1$ the pairing:

$$H_c^i(X, F) \times \mathrm{Ext}_X^{2-i}(F, \mu_n) \to H_c^2(X, \mu_n) \qquad \text{(Pairing 3.1)}$$

is perfect. We need some dÃľvissage lemmas:

Lemma 3.2.4. *Let X be a smooth curve on an algebraically closed field K, let $\pi : X' \to X$ be an Ãľtale map, let F' be a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X'. Then Pairing 3.1 relative to F' is perfect on X' if and only if it is perfect on X relative to $\pi_! F'$*

Proof. Since π is Ãľtale, $\pi_!$ is exact. Let $\overline{X}$ and $\overline{X'}$ be compactifiation of X and X' respectively, such that $j\pi = \pi' j'$ and π' is proper, so

$$H_c^r(X, \pi_! F') = \mathrm{Ext}_{\overline{X}}^r(\mathbb{Z}, (j\pi)_! F') = \mathrm{Ext}_{\overline{X}}^r(\mathbb{Z}, \pi'_* j'_! F') = H_c^r(X', F')$$

and

$$\mathrm{Ext}_X^r(\pi_! F', \mu_n) = \mathrm{Ext}_{X'}^r(F', \mu_n)$$

□

Lemma 3.2.5. *Pairing 3.1 is perfect if F is skyscraper, i.e. has support in a finite closed subset*

Proof. If F is skyscraper, then it is the direct sum of sheaves with support in one closed point, hence it is enough to consider the case when $F = i_* M$ where $i : Spec(k) \to X$ and M is a finite $\mathbb{Z}/m\mathbb{Z}$-module.
Since X is integral and smooth, consider the Nagata closure $X \hookrightarrow \overline{X}$, then $\overline{X}$ is proper. So consider $\overline{X} \hookrightarrow \widetilde{X}$ its normalization, and since $\overline{X}$ is proper, $\widetilde{X}$ is an integral proper smooth curve, hence smooth and projective ([Sta, 0A27]). So we have an open immersion

$$X \hookrightarrow \widetilde{X}$$

with $\widetilde{X}$ projective, so by lemma 3.2.4 we can suppose X an integral projective smooth curve, hence $Tr_{X/k} : H^2(X, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$ is an isomorphism.
So since i_* is exact:

$$H^r(X, i_* M) \cong H^r(Spec(k), M) = \begin{cases} M & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$$

and one can see ([Fu11, 8.3.6]) that $Ri^! F = F(-1)[-2]$ for any constant sheaf F, so:

$$\begin{aligned} \mathrm{Ext}^{2-r}(i_* M, \mu_n) &\cong \mathrm{Hom}_{D(X, \mathbb{Z}/n\mathbb{Z})}(i_* M, \mu_n[2-r]) \cong \mathrm{Hom}_{D(\mathbb{Z}/n\mathbb{Z})}(M, Ri^! \mathbb{Z}/n\mathbb{Z}(1)[2-r]) \\ &\cong \mathrm{Hom}_{D(\mathbb{Z}/n\mathbb{Z})}(M, \mathbb{Z}/n\mathbb{Z}(1)[-r])) = \begin{cases} \mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z}) & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Recall that the pairing

$$M \times \mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

is perfect for Pontryagin duality since M is finite. So the pairing

$$H^0(x, M) \times Ext^2(M, Ri^! \mu_n) \to H^2(x, Ri^! \mu_n) \cong H^2(X, i_* Ri^! \mu_n) \cong H^2_x(X, \mu_n)$$

is perfect. Then since $X \setminus \{x\}$ is affine, $H^2(X \setminus \{x\}, \mu_n) = 0$, so the canonical morphism

$$H^2_x(X, \mu_n) \to H^2(X, \mu_n)$$

is epi, and since they are both free of rank 1, it is an isomorphism, so the pairing is perfect. □

Lemma 3.2.6. *Pairing 3.1 is perfect if $F = \mathbb{Z}/n\mathbb{Z}$.*

Proof. It is again possible to suppose X irreducible with a smooth projective closure $j : X \to \overline{X}$ and a closed immersion $i : \overline{X} \setminus X \to \overline{X}$ with $\overline{X} \setminus X$ finite. Then it is enough to show that

$$H^r(\overline{X}, j_! \mathbb{Z}/n\mathbb{Z}) \times Ext^2(j_! \mathbb{Z}/n\mathbb{Z}, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$$

is perfect. Since $j^* \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}_X$ and $i^* \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}_{\overline{X} \setminus X}$, we have an exact sequence

$$0 \to j_! \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to i_* \mathbb{Z}/n\mathbb{Z} \to 0$$

So for any $\mathbb{Z}/n\mathbb{Z}$-module consider the dual

$$M^D := \mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z})$$

Since $\mathbb{Z}/n\mathbb{Z}$ is injective, $(\_)^D$ is exact. So the pairing induces a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(\overline{X}, j_!\mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "(1)"] & H^r(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "(2)"] & H^r(\overline{X}, i_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "(3)"] & \cdots \\
\cdots \arrow[r] & \mathrm{Ext}^{2-r}(j_!\mathbb{Z}/n\mathbb{Z}, \mu_n)^D \arrow[r] & \mathrm{Ext}^{2-r}(\mathbb{Z}/n\mathbb{Z}, \mu_n)^D \arrow[r] & \mathrm{Ext}^{2-r}(i_*\mathbb{Z}/n\mathbb{Z}, \mu_n)^D \arrow[r] & \cdots
\end{tikzcd}$$

So by the previous lemma (3) is an isomorphism since $i_*\mathbb{Z}/n\mathbb{Z}$ has finite support, so it is enough to show that (2) is an isomorphism, hence we are reduced to the case where X is projective.
Since $\mathrm{Ext}^{2-r}(\mathbb{Z}/n\mathbb{Z}, \mu_n) = H^{2-r}(X, \mu_n)$, we have that [Del, Arcata 3.5]

$$\mathrm{Ext}^{2-r}(\mathbb{Z}/n\mathbb{Z}, \mu_n) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } r = 0 \\ Pic^0(X)_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } r = 1 \\ \mu_n \cong \mathbb{Z}/n\mathbb{Z} & \text{if } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

So they have the same number of elements, hence it is enough to show that 2 is injective
The pairing for $r = 0$ is given by $(\phi, \psi) \mapsto \psi\phi$ in

$$\mathrm{Hom}_{D(X,\mathbb{Z}/n\mathbb{Z})}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{Hom}_{D(X,\mathbb{Z}/n\mathbb{Z})}(\mathbb{Z}/n\mathbb{Z}, \mu_n[2]) \to \mathrm{Hom}_{D(X,\mathbb{Z}/n\mathbb{Z})}(\mathbb{Z}/n\mathbb{Z}, \mu_n[2])$$

So if $\psi \in Ker(2)$, then, for any ϕ, $\psi\phi = 0$, hence $\psi = 0$, so for $r = 0$ (2) is an iso, and the same argument works for $r = 2$ on ϕ choosing an isomorphism $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$.
So it remains $r = 1$. For any $\alpha \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ consider the associated torsor $\pi : X' \to X$, with π Galois and finite Ãľtale. Then the image of α into $H^1(X', \mathbb{Z}/n\mathbb{Z})$ is zero, hence

$$\alpha \in Ker(H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X, \pi_*\mathbb{Z}/n\mathbb{Z}))$$

and since π is surjective, $\mathbb{Z}/n\mathbb{Z} \to \pi_*\mathbb{Z}/n\mathbb{Z}$ is injective, hence if F is the cokernel we have an exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \pi_*\mathbb{Z}/n\mathbb{Z} \to F \to 0$$

which induces a morphism of long exact sequences

$$\begin{tikzcd}
H^r(X, \mathbb{Z}/n\mathbb{Z}) \arrow[r, "v"] \arrow[d, "a"] & H^r(X, \pi_*\mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "b"] & H^r(X, F) \arrow[r, "\partial"] \arrow[d, "c"] & \cdots \\
\mathrm{Ext}^{2-r}(\mathbb{Z}/n\mathbb{Z}, \mu_n)^* \arrow[r] & \mathrm{Ext}^{2-r}(\pi_*\mathbb{Z}/n\mathbb{Z}, \mu_n)^* \arrow[r] & \mathrm{Ext}^{2-r}(F, \mu_n)^* \arrow[r, "\partial'"] & \cdots
\end{tikzcd}$$

We have already shown that a^0 is an iso, hence for lemma 3.2.4 b^0 is also an iso.
Notice that $\alpha \in Ker(v^1) = Im(\partial^0)$, so if $a^1(\alpha) = 0$, there is a lift β such that $\partial'^0 c^0(\beta) = 0$, so there is $\gamma \in H^0(X, \pi_*\mathbb{Z}/n\mathbb{Z})$ such that $c^0 v^0(\gamma) = c^0(\beta)$. So to conclude we need to show that c^0 is mono, since if $\beta = v^0(\gamma)$, then $\alpha = 0$.
Since π is finite Ãľtale, $\pi_*\mathbb{Z}/n\mathbb{Z}$ is lcc, so F is lcc. It can be shown [Fu11, 5.8.1] that there is

a surjective finite Ãľtale morphism $\pi' : X'' \to X$ such that π'^*F is constant, and since every $\mathbb{Z}/n\mathbb{Z}$-module of finite type admits an injection into a free $\mathbb{Z}/n\mathbb{Z}$-module, consider a mono $\pi'^*F \to L$. Hence since π is surjective $F \to \pi'_*\pi'^*F$ is mono, hence for some G there is an exact sequence

$$0 \to F \to \pi'_*L$$

which induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,F) \arrow[r] \arrow[d, "c^0"] & H^0(X,\pi'_*L) \arrow[d, "(\star)"] \\
& \mathrm{Ext}^2(F,\mu_n)^* \arrow[r] & \mathrm{Ext}^2(\pi'_*L,\mu_n)^*
\end{tikzcd}$$

So by lemma 3.2.4, $(\star)$ is an isomorphism, hence we conclude. □

Theorem 3.2.7. *Pairing 3.1 is perfect for any constructible sheaf F*

Proof. It can be shown [Fu11, 5.8.5] that there exists an Ãľtale morphism of finite type $f : U \to X$ such that $f_!\mathbb{Z}/n\mathbb{Z} \to F$ is surjective, let G be its kernel, which is again constructible. Then we have a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^r(F,\mu_n) \arrow[r] \arrow[d, "(1)"] & \mathrm{Ext}^r(f_!\mathbb{Z}/n\mathbb{Z},\mu_n) \arrow[r] \arrow[d, "(2)"] & \mathrm{Ext}^r(G,\mu_n) \arrow[r] \arrow[d, "(3)"] & \cdots \\
\cdots \arrow[r] & H_c^{2-r}(X,F)^* \arrow[r] & H_c^{2-r}(X,f_!\mathbb{Z}/n\mathbb{Z})^* \arrow[r] & H_c^{2-r}(X,G)^* \arrow[r] & \cdots
\end{tikzcd}$$

For lemma 3.2.4 and 3.2.6 (2) is an iso for any r. So for $r = 0$ (1) is an mono. This is true for any constructible sheaf, so for $r = 0$, (3) is a mono, so (1) is an iso. This is true for any constructible sheaf, so for $r = 0$, (3) is an iso. We conclude applying the same argument for all r □

Corollary 3.2.8 (PoincarÃľ Duality for curves). *With F locally constant constructible, let $F^D = \mathcal{H}om(F,\mu_n)$ we have a perfect pairing*

$$H_c^i(X,F) \times H^{2N-i}(X,F^D) \to \mathbb{Z}/n\mathbb{Z}$$

Lemma 3.2.9. *Let X be a regular scheme of pure dimension 1, $j : U \hookrightarrow X$ an open immersion, Λ a Noetherian ring with $n\Lambda = 0$ and such that Λ is an injective Λ-module. Then for every F lcc on U*

$$\mathcal{H}om(j_*F,\Lambda) = j_*\mathcal{H}om(F,\Lambda)$$

and for every $q > 0$

$$\mathcal{E}xt^q(j_*F,\Lambda) = 0$$

Proof. [Fu11]

Theorem 3.2.10. *Let X be a smooth curve over an algebraically closed field, $j : U \hookrightarrow X$ a dominant open immersion, F a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on U, $F^D = \mathcal{H}om(F, \mu_n)$. Then we have a perfect pairing*

$$H^i_c(X, j_*F) \times H^{2N-i}(X, j_*F^D) \to \mathbb{Z}/n\mathbb{Z}$$

Proof. By the previous lemma the spectral sequence

$$H^p(X, \mathcal{E}xt^q(j_*F, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{Ext}^{p+q}(j_*F, \mathbb{Z}/n\mathbb{Z})$$

degenerates in degree 2 and

$$\mathcal{H}om(j_*F, \mathbb{Z}/n\mathbb{Z}) \cong j_*F^D$$

So $\mathrm{Ext}^p(j_*F, \mathbb{Z}/n\mathbb{Z}) = H^p(X, j_*F^D)$ and the result follows from corollary 3.2.8 applied to j_*F □

3.2.2 Finite fields

Let now k be a finite field of characteristic p, X a smooth curve over k.
Recall that for any Galois covering $Y \xrightarrow{\pi} X$ with Galois group G, the Ext spectral sequence gives a quasi isomorphism

$$R\Gamma(X, F) \cong R\Gamma(G, R\Gamma(Y, \pi^*F))$$

In particular, if we consider the separable (hence, algebraic) closure $\overline{k}$ of k, the normalization $\overline{X} \to X$ is a Galois covering with Galois group $G_k \cong \widehat{\mathbb{Z}}$. Hence we have a spectral sequence

$$H^p(\widehat{\mathbb{Z}}, H^q(\overline{X}, F) \Rightarrow H^{p+q}(X, F)$$

So if F is constructible $H^q(\overline{X}, F)$ is finite and we have that if M is a finite G_k-module, we have that if $\wp = Fr - id$, where Fr is the Frobenius who generates G_k, then

$$H^r(G_k, M) = \begin{cases} {}_\wp M = Ker(\wp) & \text{if } r = 0 \\ M_\wp = CoKer(\wp) & \text{if } r = 1 \\ 0 & \text{otherwise} \end{cases}$$

So if F is constructible the spectral sequence is a two-columns, hence we have exact sequences

$$0 \to H^1(G_k, H^{n-1}(\overline{X}, \pi^*F)) \to H^n(X, F) \to H^0(G_k, H^n(\overline{X}, \pi^*F)) \to 0$$

And by replacing X with its Nagata compactifiation and F by $j_!F$ we have the same for compact supported:

$$0 \to H^1(G_k, H^{n-1}_c(\overline{X}, \pi^*F)) \to H^n_c(X, F) \to H^0(G_k, H^n_c(\overline{X}, \pi^*F)) \to 0$$

So in particular, since $H^3_c(\overline{X}, \mu_n) = 0$ we have an iso $(H^2_c(\overline{X}, \mu_n))_\wp \cong H^3_c(X, \mu_n)$, but since

$$H^2_c(\overline{X}, \mu_n) \cong Pic(X)/nPic(X) \xrightarrow{deg} \mathbb{Z}/n\mathbb{Z}$$

And the Frobenius does not change the degree of a divisor, in this case $Fr = Id$ and $(H_c^2(\overline{X}, \mu_n))_\wp \cong H_c^2(\overline{X}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$, hence

$$H_c^3(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

So by dualizing and taking $F^D = \mathcal{H}om(F, \mu_n)$ we have a morphism of short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_c^{n-1}(\overline{X}, \pi^*F)_\wp \arrow[r] \arrow[d, "(1)"] & H_c^n(X, F) \arrow[r] \arrow[d, "(2)"] & H_c^n(\overline{X}, \pi^*F)_\wp \arrow[r] \arrow[d, "(3)"] & 0 \\
0 \arrow[r] & ({}_\wp H^{3-r}(\overline{X}(\pi^*F^D)^* \arrow[r] & H^{3-r}(X, F^D)^* \arrow[r] & H^{2-r}(\overline{X}, \pi^*F^D)_\wp)^* \arrow[r] & 0
\end{tikzcd}$$

So to prove that it is an isomorphism, we need to rove that (2) and (3) are.
Considering Tate duality for finite fields (theorem 1.1.9): for any finite G_k-module M, if $M^\bigstar = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ we have an isomorphism of finite abelian groups:

$$({}_\wp H^{3-r}(\overline{X}, \pi^*F^D))^\bigstar \cong (H^{3-r}(\overline{X}, \pi^*F^D)^\bigstar)_\wp$$

and since F^D is finite annihilated by n, this gives an isomorphism

$$({}_\wp H^{3-r}(\overline{X}, \pi^*F^D))^* \cong (H^{3-r}(\overline{X}, \pi^*F^D)^*)_\wp$$

Hence (1) is an isomorphism since it is the kernel of the isomorphism of PoincarÃľ duality. With the same idea, (3) is an isomorphism, hence we have

Theorem 3.2.11. *If k is a finite field, X/k is a smooth curve, n is invertible on X, then for any constructible sheaf F we have a perfect pairing*

$$H_c^r(X, F) \times Ext_X^{3-r}(F, \mu_n) \to H_c^3(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

Corollary 3.2.12. *If F is locally constant constructible, $F^D := \mathcal{H}om(F, \mu_n)$ then we have*

$$Ext_X^r(F, \mu_n) \cong H^r(X, F^D)$$

So PoincarÃľ duality gives a perfect pairing

$$H_c^r(X, F) \times H^{3-r}(X, F^D) \to H_c^3(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

Proof. Same as for algebraically closed field using lemma 3.2.2

Remark 3.2.13. If F is killed by n and F' is n-divisible, then if we take $F' \to I^\bullet$ an injective resolution of abelian groups, then ${}_mF' \to I^\bullet$ is an injective resolution $\mathbb{Z}/m\mathbb{Z}$-modules of ${}_mF'$, since $F'/{}_mF' \cong F'$ and I^r is divisible by all n prime to n, hence

$$\mathrm{Ext}^r_{Sh(Y,\mathbb{Z}/n\mathbb{Z})}(F, {}_mF') \cong H^r(Hom_{Sh(Y,\mathbb{Z}/n\mathbb{Z})}(F, I^\bullet)) \cong H^r(Hom_{Sh(Y)}(F, I^\bullet)) = \mathrm{Ext}^r(F, F')$$

In particular, if F is killed by m, then

$$\mathrm{Ext}^r_X(F, \mathbb{G}_m) \cong \mathrm{Ext}^r_{Sh(X,\mathbb{Z}/n\mathbb{Z})}(F, \mu_n)$$

So in the following chapters we will consider $\mathbb{G}_m$ as a dualizing sheaf for generalize this.

Chapter 4

Arithmetics: Artin-Verdier duality

4.1 Local Artin-Verdier duality

Let's keep the notation from Section B.12 From now on, $\mathcal{O}$ would be an henselian DVR with finite residue field

Recall that if $X = Spec(\mathcal{O})$, then for all Ã l'tale sheaves F

$$H^p(X, F) = \mathrm{Ext}^p_{S_{\mathcal{O}}}(\mathbb{Z}, F)$$

Since $\Gamma_{\mathcal{O}}(F) = Hom_{Ab}(\mathbb{Z}, \Gamma_{\mathcal{O}}(F)) = Hom_{Sh_{Ab}(X_{et})}(\mathbb{Z}_X, F) = Hom_{S_{\mathcal{O}}}(\mathbb{Z}, F)$.

We can define by the same idea the cohomology with support in the closed point:

$$H^p_x(X, F) = \mathrm{Ext}^p_{S_{\mathcal{O}}}(i_*\mathbb{Z}, F)$$

Proposition 4.1.1. *The cohomology of $j_*\mathbb{G}_{mK}$ on X is computed as follows*

$$H^p(X, j_*\mathbb{G}_{mK}) = H^p(G_K, \overline{K}^*) = \begin{cases} K^* & \text{if } p = 0 \\ \mathbb{Q}/\mathbb{Z} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since $\Gamma_{S_K} = \Gamma_{S_{\mathcal{O}}} j_*$, we have a spectral sequence

$$H^p(X, R^q j_* F) \Rightarrow H^{p+q}(Spec(K), F)$$

And since if $F = \mathbb{G}_{mK}$ we have for remark B.12.1 $R^q(j_*\mathbb{G}_{mK}) = 0$ for $q > 0$, it degenerates in degree 2, hence

$$H^p(X, j_*\mathbb{G}_{mK}) = H^p(Spec(K), \mathbb{G}_{mK}) = H^p(G_K, \overline{K}^*)$$

Proposition 4.1.2. *For any $N \in S_k$ we have*

$$H^p(X, i_*N) = H^p_x(X, i_*N)$$

And the cohomology of $i_*\mathbb{Z}$ on X is computed as follows

$$H^p(X, i_*\mathbb{Z}) = H^p_x(X, i_*\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Q}/\mathbb{Z} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Consider the short exact sequence:

$$0 \to j_! \mathbb{Z} \to \mathbb{Z} \to i_* \mathbb{Z} \to 0$$

We have that i_* and $j_!$ are exact and preserve injectives, so $R\mathrm{Hom}_{S_{\mathcal{O}}}(j_!\mathbb{Z}, i_*N) = R\mathrm{Hom}_{S_K}(\mathbb{Z}, j^* i_* N) = 0$. From the long exact sequence of $R\mathrm{Hom}(\_, i_*N)$ we get the first equality.
Since i_* is fully faithful, exact and preserves injectives, we have

$$R\Gamma_x(X, i_*\mathbb{Z}) = R\mathrm{Hom}_{S_{\mathcal{O}}}(i_*\mathbb{Z}, i_*\mathbb{Z}) \cong R\mathrm{Hom}_{S_k}(\mathbb{Z}, \mathbb{Z}) = R\Gamma(x, \mathbb{Z})$$

In particular $H^q_x(X, i_*\mathbb{Z}) = H^q(G_k, \mathbb{Z})$, which gives the result.

Proposition 4.1.3. *Combining the previous results, we get*

a)

$$H^p(X, \mathbb{G}_{m\mathcal{O}}) = \begin{cases} \mathcal{O}^* & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

b)

$$H^p_x(X, \mathbb{G}_{m\mathcal{O}}) = \begin{cases} \mathbb{Z} & \text{if } p = 1 \\ \mathbb{Q}/\mathbb{Z} & \text{if } p = 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. a) Apply proposition 4.1.1 and proposition 4.1.2 to

$$0 \to \mathbb{G}_{m\mathcal{O}} \to j_*\mathbb{G}_{mK} \to i_*\mathbb{Z} \to 0$$

in degree 0 we have $K_0^* \to \mathbb{Z}$, in degree 2 $id : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$

b) Since by remark B.12.1 we have $Rj_*\mathbb{G}_m = j_*\mathbb{G}_m$, we have that $\mathrm{Hom}_{D(S_{\mathcal{O}})}(i_*\mathbb{Z}, j_*\mathbb{G}_m) = \mathrm{Hom}_{D(S_K)}(j^*i_*\mathbb{Z}, \mathbb{G}_m) = 0$, so by applying $\mathrm{Hom}_{D(S_{\mathcal{O}})}(i_*\mathbb{Z}, \_)$ to the previous exact sequence, since i_* is fully faithful, exact and preserves injectives we have an isomorphism

$$\mathrm{Ext}^p_{S_k}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Ext}^{p+1}_{S_{\mathcal{O}}}(i_*\mathbb{Z}, \mathbb{G}_{m\mathcal{O}}) = H^{p+1}_x(X, \mathbb{G}_{m\mathcal{O}})$$

And we already computed $\mathrm{Ext}^p(\mathbb{Z}, \mathbb{Z}) = H^p(G_k, \mathbb{Z})$

□

So we can now consider the pairing

$$\mathrm{Ext}^r(F, \mathbb{G}_{m\mathcal{O}}) \times H^{3-r}_x(X, F) \to H^3_x(X, \mathbb{G}_{m\mathcal{O}}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z} \qquad (\bigstar)$$

given by the cup-product on the derived category

$$\mathrm{Hom}_{D(S_{\mathcal{O}})}(F, \mathbb{G}_{m\mathcal{O}}[r]) \times \mathrm{Hom}_{D(S_{\mathcal{O}})}(i_*\mathbb{Z}, F[3-r]) \to \mathrm{Hom}_{D(S_{\mathcal{O}})}(i_*\mathbb{Z}, \mathbb{G}_m[3])$$

$$f \cup g \mapsto f \circ g[3-r]$$

and the maps

$$\alpha^i(X, F) : \mathrm{Ext}^r(F, \mathbb{G}_m) \to H^{3-r}_x(X, F)^*$$

where M^* is the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

Definition 4.1.4. If X is a scheme of dimension 1, Λ a ring, then a sheaf F is constructible (resp. Λ-constructible) if there exists a dense open U such that:

(a) F is locally constant defined by a finite abelian group (resp. finitely generated Λ-module)

(b) for all $x \notin U$, $F_{\bar{x}}$ is a finite abelian group (resp. finitely generated Λ-module)

To see the equivalence with the definition given in Section 2.2.1, we have the following:

Proposition 4.1.5. *If X is a Noetherian scheme, Λ a Noetherian ring F a sheaf, then F is constructible (resp. Λ-constructible) if and only if for any irreducible closed subset Y of X there is a nonempty open subset V of Y such that F_V is locally constant constructible (resp. locally constant Λ-constructible).*

Proof. see [Fu11, Proposition 5.8.3]

Remark 4.1.6. By definition, if X is a trait, then F is constructible (resp. $\mathbb{Z}$-constructible) if and only if j^*F and i^*F are finite Galois modules (resp. of finite type).

Let now $p = char(K)$ (could be 0!)

Theorem 4.1.7 (Local Artin-Verdier Duality). *If F is a $\mathbb{Z}$-constructible sheaf without p-torsion, then*

(a) *(i)* $\alpha^0(X, F)$ *defines an isomorphism*

$$Hom_{S_{\mathcal{O}}}(F, \mathbb{G}_m)^{\wedge} \to H^3_x(X, F)^*$$

(ii) $Ext^1_{S_{\mathcal{O}}}(F, \mathbb{G}_m)$ *is finitely generated and* $\alpha^1(X, F)$ *defines an isomorphism*

$$Ext^1_{S_{\mathcal{O}}}(F, \mathbb{G}_m)^{\wedge} \to H^2_x(X, F)^*$$

(iii) For $r \geq 2$ $Ext^r_{S_{\mathcal{O}}}(F, \mathbb{G}_m)$ are torsion of cofinite type (i.e. duals of groups of finite type), and $\alpha^r(X, F)$ is an isomorphism

(b) If F is constructible such that $pF = F$, then (★) is a perfect pairing and and all the groups involved are finite.

Proof. Consider the map

$$\alpha^r : H^r_x(X, F) \to \mathrm{Ext}^{3-r}(F, \mathbb{G}_{m\mathcal{O}})^*$$

In particular, α^r is defined by a morphism of δ-functors $D^b(X, \mathbb{Z}) \to D(Ab)$[1]

$$\mathrm{Hom}_{D(X,\mathbb{Z})}(i_*\mathbb{Z}[-r], \_) \to \mathrm{Hom}_{D(X,\mathbb{Z})}(\_, \mathbb{G}_{m\mathcal{O}}[3-r])^*$$

So if

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

[1]The dual passes to the derived category of abelian groups since $\mathbb{Q}/\mathbb{Z}$ is divisible

is an exact sequence and $\alpha^r_{F_i}$ is an isomorphism for two of the elements of the exact sequence, then it is an isomorphism also on the third one by TR3 (see Chapter C). So since for all Ãľtale sheaves we have the exact sequence

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

it is enough to prove the theorem for $j_! M$ and $i_* N$, with $M \in S_K$ and $N \in S_k$. We want to reduce to the case of Tate local duality:

$i_* N$ Consider again the exact sequence

$$0 \to \mathbb{G}_{m\mathcal{O}} \to j_* \mathbb{G}_{mK} \to i_* \mathbb{Z} \to 0$$

so by the same idea as before, applying $\mathrm{Hom}_{D(S_\mathcal{O})}(i_* N, \_)$ we have

$$\mathrm{Ext}^p_{S_k}(N, \mathbb{Z}) \cong \mathrm{Ext}^{p+1}_{S_\mathcal{O}}(i^* N, \mathbb{G}_{m\mathcal{O}})$$

And again $H^r_x(X, \mathbb{G}_{m\mathcal{O}}) = H^{r-1}(G_k, \mathbb{Z})$, so the duality translates in Tate duality for the finite field k:

$$H^r(G_k, N) \times \mathrm{Ext}^{2-r}_{S_k}(N, \mathbb{Z}) \to H^2(G_k, \mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

So $\mathrm{Ext}^1_{S_\mathcal{O}}(i_* N, \mathbb{G}_m)$ is finitely generated and $\widehat{\alpha}^1 : \mathrm{Ext}^1_{S_\mathcal{O}}(i_* N, \mathbb{G}_m)^\wedge \to H^2_x(X, i_* N)^*$ is an isomorphism, α^2 is an isomorphism of finite groups and α^3 is an isomorphism of groups of cofinite type. For $r > 3$ the groups involved are all zero.

$j_! M$ Consider the exact sequence for the cohomology with support

$$H^r_x(X, F) \to H^r(X, F) \to H^r(G_K, j^* F) \to$$

Then if $F = j_! M$, we have that $\Gamma(X, j_!(\_))$ is the zero functor since again we have the exact sequence

$$0 \to j_! M \to R j_* M \to i_* i^* R j_* M \to 0$$

which induces in degree 0

$$0 \to \Gamma(X, j_! M) \to M^{G_K} \xrightarrow{\sim} (M^{G_{in}})^{G_k}$$

and $j_!$ is exact. If $j_!$ sends injectives to acyclics, we can derive and get $R\Gamma(X, j_!(\_)) = 0$.
To prove this, take I injective and consider the exact sequence

$$0 \to j_! I = (I, 0, 0) \to j_* I = (I, \tau I, id) \to i_* i^* j_* I = (0, \tau I, 0) \to 0$$

It is an injective resolution of $j_! I$ since i^*, i_* and j_* preserves injectives, so $\mathrm{Ext}^q(\mathbb{Z}, j_! I) = 0$ for $q > 1$ and applying $\mathrm{Hom}(\mathbb{Z}, \_)$ we get

$$0 \to \mathrm{Hom}_{S_\mathcal{O}}(\mathbb{Z}, j_! I) \to \mathrm{Hom}_{S_\mathcal{O}}(\mathbb{Z}, j_* I) = I^{G_K} \to \mathrm{Hom}_{S_\mathcal{O}}(\mathbb{Z}, i_* i^* j_* I) \cong (I^{G_I})^{G_k}$$

And since $(I^{G_I})^{G_k} = I^{G_K}$ we have also $\mathrm{Ext}^1(\mathbb{Z}, j_! I) = 0$, hence $j_! I$ is acyclic.
So we have that $H^p_x(X, j_! M) \cong H^{p-1}(G_K, M)$ Moreover, we have that $j_! \dashv j^*$ are exact, hence

$$\mathrm{Hom}_{D(S_{\mathcal{O}})}(j_! M, \mathbb{G}_{m\mathcal{O}}) \cong \mathrm{Hom}_{D(S_K)}(M, \overline{K}^*)$$

So we reduce to Tate duality for the henselian field K:

$$H^r(G_K, M) \times \mathrm{Ext}^{2-r}_{G_K}(M, \overline{K}^\times) \to H^2(G_K, \overline{K}^*) = \mathbb{Q}/\mathbb{Z}$$

So now $\mathrm{Hom}_{S_{\mathcal{O}}}(j_! M, \mathbb{G}_m)$ is finitely generated and $\widehat{\alpha}^0 : \mathrm{Hom}_{S_{\mathcal{O}}}(, j_! M)^\wedge \to H^3_x(X, j_! M)^*$ is an isomorphism, α^1 is an isomorphism of finite groups and α^2 is an isomorphism of groups of cofinite type. For $r > 2$ the groups involved are all zero.

Hence, by using the exact sequence, we have for $r \geq 2$

$$\begin{tikzcd}
\mathrm{Ext}^r_{S_{\mathcal{O}}}(j_! j^* F, \mathbb{G}_m) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^r_{S_{\mathcal{O}}}(F, \mathbb{G}_m) \arrow[r] \arrow[d] & \mathrm{Ext}^r_{S_{\mathcal{O}}}(i_* i^* F, \mathbb{G}_m) \arrow[d, "\sim"] \\
H^{3-r}_x(X, j_! j^* F)^* \arrow[r] & H^{3-r}_x(X, F)^* \arrow[r] & H^{3-r}_x(X, i_* i^* F)^*
\end{tikzcd}$$

So we deduce the result for $r \geq 2$. For $r = 1$, since $\mathrm{Ext}^1_{S_{\mathcal{O}}}(i_* i^* F, \mathbb{G}_m)$ is finite we have

$$\begin{tikzcd}
\mathrm{Ext}^1_{S_{\mathcal{O}}}(j_! j^* F, \mathbb{G}_m)^\wedge \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1_{S_{\mathcal{O}}}(F, \mathbb{G}_m)^\wedge \arrow[r] \arrow[d] & \mathrm{Ext}^1_{S_{\mathcal{O}}}(i_* i^* F, \mathbb{G}_m) \arrow[d, "\sim"] \\
H^2_x(X, j_! j^* F)^* \arrow[r] & H^2_x(X, F)^* \arrow[r] & H^2_x(X, i_* i^* F)^*
\end{tikzcd}$$

And finally for $r = 0$ we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{S_{\mathcal{O}}}(F, \mathbb{G}_m)^\wedge \arrow[r] \arrow[d] & \mathrm{Hom}_{S_{\mathcal{O}}}(i_* i^* F, \mathbb{G}_m)^\wedge \arrow[d, "\sim"] \\
0 \arrow[r] & H^3_x(X, F)^* \arrow[r] & H^3_x(X, i_* i^* F)^*
\end{tikzcd}$$

And since for F constructible without p torsion all the groups involved in Tate duality are finite, we are done. □

Corollary 4.1.8. *If F is lcc such that $pF = F$, then consider $F^D = \mathcal{H}om(F, \mathbb{G}_m)$ the Cartier dual, then we have a pairing*

$$H^r_x(X, F^D) \times H^{3-r}_x(X, F) \to H^3_x(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

Proof. Since we have

$$\mathrm{Hom}_{D(X)}(\mathbb{Z}, R\mathcal{H}om(F, \mathbb{G}_m)) \cong \mathrm{Hom}_{D(X)}(\mathbb{Z} \otimes^{\mathbb{L}} F, \mathbb{G}_m) = \mathrm{Hom}_{D(X)}(F, \mathbb{G}_m)$$

We just need to show that $R\mathcal{H}om(F, \mathbb{G}_m) = \mathcal{H}om(F, \mathbb{G}_m)$[2].
We have that, since the stalk is an exact functor,

$$R\mathcal{H}om(F, \mathbb{G}_m)_{\bar{x}} = \mathrm{RHom}_{\mathbb{Z}}(F_{\bar{x}}, (\mathbb{G}_m)_{\bar{x}}) = R\mathrm{Hom}_{\mathbb{Z}}(F_{\bar{x}}, \mathcal{O}^{un\times})$$

and since $\mathcal{O}^{un\times}$ is divisible by all the primes that divide $F_{\bar{x}}$, we have $R\mathrm{Hom}_{\mathbb{Z}}(F_{\bar{x}}, \mathcal{O}^{un\times}) = \mathrm{Hom}_{\mathbb{Z}}(F_{\bar{x}}, \mathcal{O}^{un\times})$.
On the other hand,

$$R\mathcal{H}om(F, \mathbb{G}_m)_{\bar{\eta}} = \mathrm{RHom}_{\mathbb{Z}}(F_{\bar{\eta}}, (\mathbb{G}_m)_{\bar{\eta}}) = R\mathrm{Hom}_{\mathbb{Z}}(F_{\bar{\eta}}, \overline{K}^{\times})$$

and we conclude for the same reason as before. □

4.2 Global Artin-Verdier duality: preliminaries

Notations:

- K will be a global field, $\overline{K}$ a fixed separable closure, G_K its absolute Galois group, $S_K = S_f \cup S_\infty$ the set of places.
- When K is a number field, $X = Spec(\mathcal{O}_K)$, when K is a function field, k will be the field of constants and X will be the unique connected integral proper smooth curve over k such that $k(X) = K$. The residue field at a nonarchimedean prime v will be denoted as $k(v)$.
- The generic point of X will be $\eta = Spec(K)$ and the canonical inclusion will be $g : \eta \to X$
- $U \subseteq X$ is an open subset and $U^0 \subseteq S_f$ is the set of places of K corresponding to the closed points of U
- If v is an archimedean place, K_v will denote the completion of K at v, and if v is archimedean, then K_v will be the fraction field of the Henselization of the local ring $\mathcal{O}_{X,v}$. G_v will denote the Galois group of K_v, with a fixed embedding we identify G_v as a subgroup of G_K We have a canonical map $Spec(K_v) \to \eta$, and if v is nonarchimedean we have a base change diagram

$$\begin{array}{ccc} Spec(K_v) & \longrightarrow & \eta \\ \downarrow & & \downarrow \\ Spec(\mathcal{O}_v^h) & \longrightarrow & X \end{array}$$

- If F is a sheaf on $U \subseteq X$, then F_v will denote the sheaf on $Spec(K_v)$ obtained by the pull back on $Spec(K_v) \to \eta \to X$
- If v is a place and F is a sheaf on $Spec(K_v)$, with corresponding Galois module M, then we will denote

$$H^r(K_v, F) := \begin{cases} \widehat{H}^r(G_v, M) & \text{if } v \text{ is archimedean} \\ H^r(G_v, M) & \text{if } v \text{ is finite} \end{cases}$$

[2] i.e. that $\mathcal{E}xt^r(F, \mathbb{G}_m) = 0$ for all $r \neq 0$

4.2.1 Cohomology of $\mathbb{G}_m$

Lemma 4.2.1. *If $g : \eta \to X$ is the generic point, then $R^s g_* \mathbb{G}_m = 0$ for all $s > 0$, i.e. $Rg_*\mathbb{G}_m = g_*\mathbb{G}_m$*

Proof. If $\bar{x}$ is a geometric point whose image correspond to the nonarchimedean place v, by using the base change we defined above

$$(R^s g_* \mathbb{G}_m)_{\bar{x}} = H^s(\eta \times_X Spec(\mathcal{O}_v^{sh}), \bar{x}'^* \mathbb{G}_m) = H^s(Spec(K_v^{sh}), \bar{x}'^* \mathbb{G}_m) = H^s(I_v, \overline{K_v}^\times) = 0, \quad s > 0$$

and if $\bar{x}$ is the geometric generic point,

$$(R^s g_* \mathbb{G}_m)_{\bar{x}} = H^s(\{1\}, \overline{K}^\times) = 0, \quad s > 0$$

$\square$

Proposition 4.2.2. *Let $U \subseteq X$, $S = S_K \setminus U^0$. Then*

$$H^0(U, \mathbb{G}_m) = \Gamma(U, \mathcal{O}_U^\times)$$
$$H^1(U, \mathbb{G}_m) = Pic(U)$$

and there is an exact sequence

$$0 \to H^2(U, \mathbb{G}_m) \to \bigoplus_{v \in S} Br(K_v) \to \mathbb{Q}/\mathbb{Z} \to H^3(U, \mathbb{G}_m) \to 0$$

And for $r \geq 4$ $H^r(U, \mathbb{G}_m) \cong \bigoplus_{v\ real} H^r(K_v, \mathbb{G}_m)$

Proof. We have the exact sequence as defined in theorem B.8.6:

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_m \to Div_U \to 0$$

And by theorem B.8.11 we have H^0 and H^1.
By the previous lemma, $H^p(U, g_*\mathbb{G}_m) = H^p(Spec(K), \mathbb{G}_m)$ And by definition

$$H^r(U, Div_U) = \bigoplus_{v \in U^0} H^r(U, i_*\mathbb{Z}) = \bigoplus_{v \in U^0} H^r(Spec(k(v)), \mathbb{Z})$$

Since $G_k(v) = \widehat{\mathbb{Z}}$, we have

$$H^r(Spec(k(v)), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Q}/\mathbb{Z} \cong Br(K_v) & \text{if } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

So the long exact sequence in cohomology gives

$$0 \to H^2(U, \mathbb{G}_m) \to Br(K) \to \bigoplus_{v \in U^0} Br(K_v) \to H^3(U, \mathbb{G}_m) \to H^3(K, \mathbb{G}_m) \to 0$$

and for $r \geq 4$ we have isomorphisms $H^r(U, \mathbb{G}_m) = H^r(Spec(\mathcal{O}_{K,S}), \mathbb{G}_m) \cong H^r(G_S, K_S^\times) = H^r(K, \mathbb{G}_m)$, and we conclude for $r \geq 4$ using a generalization of theorem 1.3.4 to tori (see [Mil06, I, 4.21]
Global class field theory provides an exact sequence

$$0 \to Br(K) \xrightarrow{f} \bigoplus_{v \in S_K} Br(K_v) \xrightarrow{\sum inv_v} \mathbb{Q}/\mathbb{Z} \to 0$$

So we have a pair of map

$$Br(K) \xrightarrow{f} \bigoplus_{v \in S_K} Br(K_v) \xrightarrow{g} \bigoplus_{v \in U^0} Br(K_v)$$

which induces the exact sequence

$$0 \to Ker(f) = H^2(U, \mathbb{G}_m) \to Br(K) \to Ker(g) = \bigoplus_{v \in S} Br(K_v) \to coker(f) = \mathbb{Q}/\mathbb{Z}$$

Hence attaching it to the previous one we have the required exact sequence □

Remark 4.2.3. If U is a proper subset, i.e. if S contains at least one nonarchimedian place, the map

$$\bigoplus_{v \in S} Br(K_v) \to \mathbb{Q}/\mathbb{Z}$$

is epi, so the result of the proposition can be generalized as

$$0 \to H^2(U, \mathbb{G}_m) \to \bigoplus_{v \in S} Br(K_v) \to \mathbb{Q}/\mathbb{Z}$$

$$H^r(U, \mathbb{G}_m) = \bigoplus_{v \in S_\infty} H^r(K_v, \mathbb{G}_m), \quad r \geq 3$$

and recall that $H^r(K_v, \mathbb{G}_m) = 0$ if r is odd.

4.2.2 Compact supported

We need to adapt the definition of compact supported Ãľtale cohomology in order to take in account the real places.
Let F be a sheaf on U. Since U is quasi-projective over an affine scheme, we have for [Mil16, III.2.17] that $\check{H}^r(U, F) = H^r(U, F)$, so we can work with the Čech complex.
There is the canonical map defined in proposition B.5.5

$$C^\bullet(F) \to (i_v)_* C^\bullet(F_v)$$

So if v is non archimedean, let $S^\bullet(M_v) \cong C^\bullet(F_v)$ be the standard complex of M_v, and if v is real, $S^\bullet(M_v)$ will be defined as the standard complete resolution of M_v, as defined in Section 1.1.1, and in any case there is a canonical map $C^\bullet(F_v) \to S^\bullet(M_v)$
Then since we have a canonical map

$$u : C^\bullet(U, F) \to \bigoplus_{v \notin U^0} C^\bullet(K_v, F_v) = \bigoplus_{v \notin U^0} S^\bullet(M_v)$$

We define $H_c^\bullet(U,F) := Cone(u)[-1]$ and $H_c^r(U,F)$ its cohomology, we have a triangle

$$(H_c(U,F), C^\bullet(U,F), \bigoplus_{v\notin U^0} S^\bullet(M_v))$$

and a long exact sequence

$$H_c^r(U,F) \to H^r(U,F) \to \bigoplus_{v\notin U^0} H^r(K_v, F_v) \to$$

By definition it is a ∂-functor.

Remark 4.2.4. If K is totally imaginary, since we have now the exact sequence

$$0 \to j_!F \to Rj_*F \to i_*i^*Rj_*F \to 0$$

we have a long exact sequence

$$H^r(X, j_!F) \to H^r(U,F) \to H^r(X\setminus U, i^*Rj_*F) = \bigoplus_{x\in X\setminus U} H^r(x, i_x^*Rj_*F)$$

And the last by excision is $\bigoplus_{v\in X\setminus U} H^r(K_v, i_v^*Rj_*F)$, and since i_v factorizes through the generic point and $(j_*I)_{\bar\eta} \cong (I)_{\bar\eta}$ for every injective I since U is a neighbourhood of η, we have $H^r(K_v, i_v^*Rj_*F) = H^r(K_v, F_v)$. So if K is totally immaginary this definition of compact supported cohomology agrees with the usual one[3].

Proposition 4.2.5. *(a) For any $i : Z \hookrightarrow U$ a closed immersion such that $i(Z) \neq U$, F a sheaf on Z, we have $H_c^r(U, i_*F) \cong H^r(Z,F)$*

(b) For any $j : V \hookrightarrow U$ open immersion, F a sheaf on V, we have $H_c^r(U, j_!F) \cong H_c^r(V,F)$

Proof. (a) Since $(i_*F)_{\bar\eta} = 0$, we have $\bigoplus_{v\notin U^0} H^r(K_v, F_v) = 0$, hence the long exact sequence gives the isomorphism

(b) Consider the exact sequence for the cohomology with support

$$H^r_{U\setminus V}(U, j_!F) \to H^r(U, j_!F) \to H^r(V,F)$$

Since by the excision

$$H^r_{U\setminus V}(U, j_!F) \cong \bigoplus_{v\in U\setminus V} H^r_v(Spec(\mathcal{O}_v^h), j_!F)$$

and since we have the exact sequence

$$H^r_v(Spec(\mathcal{O}_v^h), j_!F) \to H^r(Spec(\mathcal{O}_v^h), j_!F) \to H^r(K_v, F_v)$$

[3] Notice that since here F is not in general torsion, the definition of proper support cohomology depends on the choice of the compactifiation!

by the vanishing of $H^r(Spec(\mathcal{O}_v^h), j_!F)$ (lemma B.12.5) we have $H_v^r(Spec(\mathcal{O}_v^h), j_!F) \cong H^{r-1}(K_v, F)$. Hence if we consider the map

$$C^\bullet(U, j_!F) \to C^\bullet(U, Rj_*j^*j_!F) = C^\bullet(V, F)$$

its mapping cone is quasi isomorphic to $\bigoplus_{v\in U\setminus V} C^\bullet(K_v, F_v)$ The cokernel of $\bigoplus_{v\notin U} S^\bullet(K_v, F_v) \to \bigoplus_{v\notin V} S^\bullet(K_v, F_v)$, since there are no archimedean places involved, is

$$\bigoplus_{v\in U\setminus V} S^\bullet(K_v, F_v) \cong \bigoplus_{v\in U\setminus V} C^\bullet(K_v, F_v)$$

So we have a sequence of triangles

$$\begin{tikzcd}
C^\bullet(U, j_!F) \arrow[r] \arrow[d] & C^\bullet(V, F) \arrow[r] \arrow[d] & \bigoplus_{v\in U\setminus V} C^\bullet(K_v, F_v) \arrow[d] \\
\bigoplus_{v\notin U} S^\bullet(K_v, F_v) \arrow[r] \arrow[d] & \bigoplus_{v\notin V} S^\bullet(K_v, F_v) \arrow[r] \arrow[d] & \bigoplus_{v\in U\setminus V} C^\bullet(K_v, F_v) \arrow[d] \\
H_c(U, j_!F) \arrow[r] & H_c(V, F) \arrow[r] & Cone(H_c(U, j_!F) \to H_c(V, F))
\end{tikzcd}$$

And since the vertical map on the right is the identity, $Cone(H_c(U, j_!F) \to H_c(V, F))$ is quasi-isomorphic to 0, hence for the long exact sequence in cohomology of the last line we get the result.

□

Corollary 4.2.6. *For every $j : V \hookrightarrow U$ open immersion, $i : U \setminus V \to U$ closed immersion, F a sheaf on U we have $H_c^r(V, F_V) \cong H_c^r(U, j_!j^*F)$ and $\bigoplus_{v\in U^0\setminus V^0} H^r(K_v, F_v) \cong H^r(U \setminus V, i^*F) \cong H_c^r(U \setminus V, i^*F)$, hence we have a long exact sequence*

$$H_c^r(V, F_V) \to H_c^r(U, F) \to \bigoplus_{v\in U\setminus V} H^r(v, F_v) \to$$

There are attempts to give a better definition of it using Artin-Verdier topology as in [Bie87] and [FM12], but right now it is known only in the case of proper schemes.
The attempt of Artin-Verdier compactifiation is in fact to express it as $R\Gamma(X, \phi_!\_)$ for some exact functor $\phi_!$.

Proposition 4.2.7. *Let $U \hookrightarrow X$ be an open immersion. Then $H_c^2(U, \mathbb{G}_m) = 0$, $H_c^3(U, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ and $H_c^r(U, \mathbb{G}_m) = 0$ for $r > 3$*

Proof. We have the exact sequences

$$0 \to H_c^2(U, \mathbb{G}_m) \to H^2(U, \mathbb{G}_m) \to \bigoplus_{v\notin U} Br(K_v) \to H_c^3(U, \mathbb{G}_m) \to H^3(U, \mathbb{G}_m) \to 0$$

and for $2r \geq 4$

$$0 \to H_c^{2r}(U, \mathbb{G}_m) \to H^{2r}(U, \mathbb{G}_m) \to \bigoplus_{v \text{ real}} H^{2r}(K_v, \mathbb{G}_m) \to H_c^{2r+1}(U, \mathbb{G}_m) \to H^{2r+1}(U, \mathbb{G}_m) \to 0$$

By remark 4.2.3 we have the result. □

Lemma 4.2.8. *For any closed immersion $i : Z \hookrightarrow U$ such that $i(Z) \neq U$, we have $H^r(Z, i^*\mathbb{G}_m) = 0$ for all $r \geq 1$.*

Proof. Since $H^r(Z, i^*\mathbb{G}_m) = \bigoplus_{v \in Z} H^r(v, i^*\mathbb{G}_m)$ hence it is enough to prove it when $Z = Spec(k(v))$ is a point. If $i : v \hookrightarrow X$ is a closed immersion, then $i^*\mathbb{G}_m$ is the g_v-module

$$(i^*\mathbb{G}_m)(Spec(\overline{k(v)})) = (\mathbb{G}_m)_v = \varinjlim_{\substack{R/\mathcal{O}_v \\ \text{unramified}}} R^\times = \mathcal{O}_v^{un\times}$$

so since $\mathcal{O}_v^{un\times}$ is g_v-cohomologically trivial, we have the result. $\square$

Remark 4.2.9. If K is a number field, we have the long exact sequence

$$0 \to H^0_c(X, \mathbb{G}_m) \to \mathcal{O}_K^\times \to \bigoplus_{v \text{ real}} K_v^\times / K_v^{\times 2} \to H^1_c(X, \mathbb{G}_m) \to Pic(X) \to 0$$

In particular,

$$H^0_c(X, \mathbb{G}_m) = \{a \in \mathcal{O}_K^\times : \sigma_v(a) > 0 \text{ for all real embeddings } \sigma_v\}$$

is the group of totally positive units, and

$$H^1_c(X, \mathbb{G}_m) = ArDiv(X)/\{a \in K^\times : \sigma_v(a) > 0 \text{ for all real embeddings } \sigma_v\}$$

Is the narrow class group (see [Nar13]).

The long exact sequence for compact supported cohomology given by the triangle

$$0 \to \mathbb{G}_m \to g * \mathbb{G}_m \to \bigoplus_{v \in X^0} (i_v)_* \mathbb{Z} \to 0$$

Is

$$H^0_c(X, g * \mathbb{G}_m) \to \bigoplus_{v \in X^0} \mathbb{Z} \to H^1_c(X, \mathbb{G}_m) \to H^1_c(X, g * \mathbb{G}_m)$$

and by the exact sequence::

$$0 \to H^0_c(X, g_* \mathbb{G}_m) \to H^0(X, g_* \mathbb{G}_m) = K^\times \to \bigoplus_{v \text{ real}} K_v^\times / K_v^{\times 2} \to H^1_c(X, g_* \mathbb{G}_m) \to 0 \text{ (Hilb 90)}$$

we deduce that $H^0_c(X, g_* \mathbb{G}_m)$ is the group of totally positive elements of $K^\times$ and since $K^\times \to \bigoplus_{v \text{ real}} K_v^\times / K_v^{\times 2}$ is epi $H^1_c(X, g_* \mathbb{G}_m) = 0$.

4.2.3 Locally constant sheaves

We generalize the ideas given in Section 1.2 to locally constant sheaves:
Throughout this subsection, U will be considered affine, and S would be the set of places not

in U. We have in the notations of Section 1.2, $G_S = \pi_1(U,\eta)$ by definition, and by definition of fundamental group we have equivalences of categories

$$\begin{tikzcd} FEt(U) \arrow[r, leftrightarrow, "(\_)\times_U \eta"] \arrow[d, leftrightarrow, "\text{Yoneda}"] & \pi_1(U,\eta) - mod^f \\ \text{L.c.c. sheaves} \arrow[ur, leftrightarrow, "(\_)_\eta"'] & \end{tikzcd}$$

Which generalizes in

$$\text{L.c. } \mathbb{Z}\text{-constructible sheaves} \xleftrightarrow{(\_)_\eta} \pi_1(U,\eta) - mod^{ft}$$

Consider the normalization $\widetilde{U}$ of U in K_S (the maximal extension of K ramified outside S), i.e. $\widetilde{U} = Spec(\mathcal{O}_S)$ by definition. Then $\widetilde{U}/U$ is the uninversal Galois covering with Galois group G_S. Notice than moreover $\pi_1(\widetilde{U},\eta) = 0$ by definition. In particular, every locally constant sheaf F on U becomes constant on $\widetilde{U}$

Proposition 4.2.10. *Let F be a l.c. $\mathbb{Z}$-constructible sheaf on U and $M = F_\eta$. Then $H^r(U,F)$ is torsion for $r > 0$ and we have*

$$H^r(U,F)(\ell) \cong H^r(G_S,M)(\ell)$$

for all ℓ invertible on U and $\ell = char(K)$.

Idea. The idea is to use the spectral sequence for the Galois cover $\widetilde{U}/U$:

$$H^r(G_S, H^s(\widetilde{U}, F_{\widetilde{U}})) \Rightarrow H^{r+s}(U,F)$$

Hence it is enough to show that $H^s(\widetilde{U}, F_{\widetilde{U}})$ is torsion and $H^s(\widetilde{U}, F_{\widetilde{U}})(\ell) = 0$ for the required ℓ.
For the base pass, $H^1(\widetilde{U}, F_{\widetilde{U}}) = \mathrm{Hom}(\pi_1(\widetilde{U},\eta), M) = 0$ since $\pi_1(\widetilde{U},\eta) = 0$.
For the general case, since $F_{\widetilde{U}}$ is constant, hence we need to consider three cases:

- $F_{\widetilde{U}} = \mathbb{Z}/\ell\mathbb{Z}$, ℓ is invertible on $\mathcal{O}_S$
  We have $F_{\widetilde{U}} \cong \mu_\ell$ and we have by Kummer exact sequence

  $$0 \to Pic(\widetilde{U}) \xrightarrow{\ell} Pic(\widetilde{U}) \to H^2(\widetilde{U}, F_{\widetilde{U}}) \to {}_\ell H^2(\widetilde{U}) \to 0$$

  And by some consideration on the groups one can show that $H^2(\widetilde{U}, F_{\widetilde{U}}) = 0$ (see [Mil06, II.2.9]). Then if we take a finite totally immaginary extension $K \subset L \subset K_S$ containing the ℓ-th roots of 1, we have for proposition 4.2.2 $H^r(U_L, \mathbb{G}_m) = 0$ since L has no real primes.

- $F_{\widetilde{U}} = \mathbb{Z}/p\mathbb{Z}$, $p = char(K)$
  We have Artin-Schreier exact sequence, and since $H^r(\widetilde{U}_{\tilde{A}rt}, \mathbb{G}_a) = H^r(\widetilde{U}_{Zar}, \mathcal{O}) = 0$ for $r > 0$.

- $F_{\widetilde{U}} = \mathbb{Z}$
  $H^r(\widetilde{U}, \mathbb{Z})$ is torision for [Mil06, II.2.10], and we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

  So from the previous points we deduce $H^r(\widetilde{U}, \mathbb{Z})(\ell) = 0$ and $H^r(\widetilde{U}, \mathbb{Z})(p) = 0$

□

From the long exact sequence

$$H_c^r(U, F) \to H^r(U, F) \to \bigoplus_{v \in S} H^r(K_v, F_v) \to$$

we deduce some nice properties (see [Mil06, II.2.11])

4.3 Global Artin-Verdier duality: the theorem

We have proved that there are trace maps $H_c^3(U, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ which commute with the restriction maps, so the cup product pairings give a pairing

$$\mathrm{Ext}_U^r(F, \mathbb{G}_m) \times H_c^{3-r}(U, F) \to \mathbb{Q}/\mathbb{Z}$$

which gives maps $\alpha^r(U, F) : \mathrm{Ext}_U^r(F, \mathbb{G}_m) \to H_c^{3-r}(U, F)^*$. The goal of this section is to prove global Artin-Verdier duality:

Theorem 4.3.1. *Let F be a $\mathbb{Z}$-constructible sheaf on an open U of X.*

(a) For $r = 0, 1$, $Ext_U^r(F, \mathbb{G}_m)$ is finitely generated and α^r induce isomorphisms:

$$Ext_U^r(F, \mathbb{G}_m)^\wedge \to H_c^{3-r}(U, F)^*$$

For $r \geq 2$, $Ext_U^r(F, \mathbb{G}_m)$ are torsion of cofinite type and α^r is an isomorphism.

(b) If F is constructible, then

$$Ext_U^r(F, \mathbb{G}_m) \times H_c^{3-r}(U, F) \to \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite abelian groups for all $r \in \mathbb{Z}$

Remark 4.3.2. If we have a triangle $0 \to F' \to F \to F'' \to 0$, and the theorem is true for two of the terms (say F and F''), the long exact sequence will imply that $\mathrm{Ext}_U^1(F', \mathbb{G}_m)$ is finitely generated, so its image in the torsion group $\mathrm{Ext}_U^2(F'', \mathbb{G}_m)$ is finite, hence the long exact sequence remains exact if we complete the first six terms, so the theorem is true also for the third one.

We will set $\widehat{\alpha}^r(U, F)$ the map we are looking for, i.e.

$$\widehat{\alpha}^r(U, F) = \begin{cases} \alpha^r(U, F)^\wedge : \mathrm{Ext}_U^r(F, \mathbb{G}_m)^\wedge \to H_c^{3-r}(U, F)^* & \text{if } r = 0, 1 \\ \alpha^r(U, F) & \text{otherwise} \end{cases}$$

Lemma 4.3.3. *Theorem 4.3.1 is true if F has support in a closed subset, i.e. if $F = i_*M$ where M is a sheaf on a closed subscheme $i : Z \to U$.*

Proof. Since $Z = \coprod_{v\in Z} v$ is a finite union of closed points, we can reduce to the case when $Z = v$ is a closed point.
We have for lemma 4.2.1, we have in $D(U)$ the exact sequence

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_m = Rg_*\mathbb{G}_m \to \bigoplus_{v\in U^0} (i_v)_*\mathbb{Z} \to 0$$

We have $\mathrm{Ext}^r_\eta(i_*F, Rg_*\mathbb{G}_m) = \mathrm{Ext}^r_\eta(g^*i_*F, \mathbb{G}_m) = 0$ and since i_u are exact functors, $\mathrm{Ext}^r_U((i_v)_*F, (i_u)_*\mathbb{Z})$
$\mathrm{Ext}^r_u((i_u)^*(i_v)_*F, \mathbb{Z}) = 0$ if $u \neq v$, so the long exact sequence gives isomorphisms

$$\mathrm{Ext}^r_U(i_*F, \mathbb{G}_m) \cong \mathrm{Ext}^{r-1}_v(F, \mathbb{Z})$$

So if M is the g_v-module corresponding to F we have for proposition 4.2.5 that $H^{3-r}_c(U, i_*F) \cong H^{3-r}(U, i_*F) \cong H^{3-r}(g_v, M)$

$$\begin{array}{ccccccc}
\mathrm{Ext}^r_U(i_*F, \mathbb{G}_m) & \times & H^{3-r}_c(U, i_*F) & \longrightarrow & H^3_c(U, \mathbb{G}_m) & \cong & \mathbb{Q}/\mathbb{Z} \\
\| & & & & & & \| \\
\mathrm{Ext}^{r-1}_{g_v}(M, \mathbb{Z}) & \times & H^{3-r}(g_v, M) & \longrightarrow & H^2(g_v, \mathbb{Z}) & \cong & \mathbb{Q}/\mathbb{Z}
\end{array}$$

So the theorem comes from Tate duality for $g_v = \widehat{\mathbb{Z}}$. □

Lemma 4.3.4. *For any $\mathbb{Z}$-constructible sheaf, $\mathrm{Ext}^r_U(F, \mathbb{G}_m)$ are of finite type for $r = 0, 1$, of cofinite type if $r = 2, 3$, and finite for $r > 3$. If F is constructible, every group is finite.*

Proof. If $F = \mathbb{Z}$, then $\mathrm{Ext}^r_U(F, \mathbb{G}_m) = H^r(U, \mathbb{G}_m)$ which have already been calculated. Using the exact sequence which defines $\mathbb{Z}/n\mathbb{Z}$, we have the theorem also for $F = \mathbb{Z}/n\mathbb{Z}$, hence for all constant $\mathbb{Z}$-constructible sheaves.
If F is locally constant $\mathbb{Z}$-constructible, there exists a Galois cover $\pi : U' \to U$ with Galois group G such that π^*F is constant associated to a G-module M (see [Fu11, Prop 5.8.1]), we have by the Ext composition with the constant sheaf $\mathbb{Z}$ (which is flat)

$$R\Gamma(G, R\mathrm{Hom}_{U'}(M, \mathbb{G}_m)) \cong R\mathrm{Hom}_U(M, \mathbb{G}_m)$$

And since $R\Gamma(G, \_) : D_{fg}(G) \to D_{fg}(G)$ and $D_{fin}(G) \to D_{fin}(G)$, the lemma for M implies the lemma for F.
Finally, if F is any $\mathbb{Z}$-constructible sheaf, let $j : V \to U$ be the open such that j^*F is locally constant and $i : U \setminus V \to U$ be the closed complement. We have the exact sequence

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

Notice that i_*i^*F has support in a finite subset, so we can use the previous lemmma and get the long exact sequence

$$\mathrm{Ext}^{r-1}_{U\setminus V}(i_*F, \mathbb{Z}) \to \mathrm{Ext}^r_U(F, \mathbb{G}_m) \to \mathrm{Ext}^r_V(j^*F, \mathbb{G}_m)$$

The lemma is true for $\mathrm{Ext}^r_V(j^*F, \mathbb{G}_m)$ since j^*F is locally constant, $\mathrm{Ext}^{r-1}_{U\setminus V}(i_*F, \mathbb{Z}) = \bigoplus_{\text{finite}} \mathrm{Ext}^{r-1}_v(F_v, \mathbb{Z}$
and for Tate duality over finite fields it is zero for $r = 0$, finitely generated for $r = 1$, finite for $r = 2$, torsion of cofinite type for $r = 3$ and zero otherwise, so the lemma follows. □

Lemma 4.3.5. *Let $j : V \to U$ be open nonempty and F a $\mathbb{Z}$-constructible sheaf on U. The theorem is true for F if and only if it is true for j^*F on V.*

Proof. We have $\mathrm{Ext}^r_U(j_!j^*F, \mathbb{G}_m) \cong \mathrm{Ext}^r_V(j^*F, \mathbb{G}_m)$ and $H^r_c(U, j_!j^*F) \cong H^r_c(V, j^*F)$ for proposition 4.2.5, hence $\widehat{\alpha}^r(U, j_!j^*F)$ can be identified with $\widehat{\alpha}^r(V, j^*F)$, and since the theorem is true on the closed complementary since it is finite, we conclude using the exact sequence that it is true on F if and only if it is true on $j_!j^*F$ if and only if it is true on j^*F □

In particular, the lemma shows that it is enough to prove the theorem for locally constant sheaves on a suitably small U.

Lemma 4.3.6. *Consider K'/K a finite Galois extension and $\pi : U' \to U$ the normalization morphism, F' a $\mathbb{Z}$-constructible sheaf on U'*

(a) There is a canonical map $Nm : \pi_*\mathbb{G}_{m\,U'} \to \mathbb{G}_{m\,U}$

(b) The composite

$$N : Ext^r_{U'}(F', \mathbb{G}_m) \to Ext^r_U(\pi_*F', \pi_*\mathbb{G}_m) \xrightarrow{Nm} Ext^r_U(\pi_*F', \mathbb{G}_m)$$

is an isomorphism

(c) $\widehat{\alpha}^r(U', F')$ is an isomoprhism if and only if $\widehat{\alpha}^r(U, \pi_*F')$ is an isomorphism.

Proof. (a) Consider $V \to U$ Ãľtale. After [Mil16, I.3.21] there is L a finite separable K-algebra such that $V \to U_L$ is an open immersion, where U_L is the normalization of U in L. By definition, if $V' = V \times_U U'$, $\Gamma(V, \pi_*\mathbb{G}_m) = \Gamma(V', \mathbb{G}_m)$. Since V' is finite over V and Ãľtale, hence normal, over U', it is the normalization of V on $K' \otimes_K L$. Hence the norm map $K' \otimes_K L \to L$ induces a unique norm map $\Gamma(V, \pi_*\mathbb{G}_m) = \Gamma(V', \mathbb{G}_m) = \mathcal{O}_{V'}(V')^\times \to \mathcal{O}_V(V)^\times$

(b) Consider $j : V \to U'$ the open subset such that $\pi j : V \to U$ is Ãľtale. Then we have $(\pi j)_! = \pi_* j_!$ and an adjunction map $(\pi j)_!(\pi j)^*G \to G$, and since we have that $j^*\pi^*\mathbb{G}_{m\,U} = \mathbb{G}_{m\,V}$, the adjunction map induces a canonical map

$$tr : (\pi j)_!\mathbb{G}_{m\,V} \to \mathbb{G}_{m\,U}$$

and since π is finite, $R\pi_* = \pi_*$ so the map passes to the derived category. So we have a canonical map

$$R\mathrm{Hom}_V(j^*F', \mathbb{G}_m) \to R\mathrm{Hom}_U((\pi j)_!j^*F', (\pi j)_!\mathbb{G}_m) \xrightarrow{tr} R\mathrm{Hom}((\pi j)_!j^*F', \mathbb{G}_m) = R\mathrm{Hom}(\pi_*j_!j^*F', \mathbb{G}_m)$$

And according to [Mil16, V, Prop 1.13] this canonical map is an isomorphism.
Since again $j^*\mathbb{G}_{m\,U'} = \mathbb{G}_{m\,V}$, we have a canonical isomorphism given by the adjunction:

$$R\mathrm{Hom}'_U(j_!j^*F', \mathbb{G}_m) \to R\mathrm{Hom}_U(j^*F', \mathbb{G}_m)$$

And the composition of this two isomorphism is N, so the theorem is true for $j_!j^*F$.
If $i : v \to U$ is a closed point, then for i_*F we have for lemma 4.3.3 that the sequence of maps is given by

$$\mathrm{Ext}^{r-1}_{\pi^{-1}(v)}(F', \mathbb{Z}) \to \mathrm{Ext}^{r-1}_v(\pi_*F', \pi_*\mathbb{Z}) \xrightarrow{Nm} \mathrm{Ext}^{r-1}_v(\pi_*F', \mathbb{Z})$$

Hence the general case follows from the triangle

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

(c) We have that since π is finite, $H^r_c(U, \pi_* F') \cong H^r_c(U', F')$, and the norm map induces $H^3(U, \pi_* \mathbb{G}_m) \xrightarrow{Nm} H^3(U, \mathbb{G}_m)$. By definition, for all $w|v \notin U$ the following diagram commutes:

$$\begin{tikzcd} Br(K'_w) \arrow[r] \arrow[d, "Nm"] & H^3_c(U', \mathbb{G}_m) \arrow[d, "Nm"] \\ Br(K_v) \arrow[r] & H^3_c(U, \mathbb{G}_m) \end{tikzcd}$$

And since

$$\begin{tikzcd} Br(K'_w) \arrow[r, "inv"] \arrow[d, "Nm"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ Br(K_v) \arrow[r, "inv"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Commutes, we have

$$\begin{tikzcd} H^3_c(U', \mathbb{G}_m) \arrow[r] \arrow[d, "Nm"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ H^3_c(U, \mathbb{G}_m) \arrow[r] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

So we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^r_{U'}(F', \mathbb{G}_m) \arrow[d, "N"] & \times & H^{3-r}_c(U', F) \arrow[r] \arrow[d, equal] & H^3_c(U, \mathbb{G}_m) \arrow[d, "Nm"] & \cong & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ \mathrm{Ext}^r_U(\pi_* F, \mathbb{G}_m) & \times & H^{3-r}_c(U, \pi_* F) \arrow[r] & H^3_c(U, \mathbb{G}_m) & \cong & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

□

Lemma 4.3.7. *(a) If F is constructible, then $H^r_c(U, F)$ is zero for $r > 3$, and if F is $\mathbb{Z}$-constructible, then it is zero for $r > 4$.*

(b) If F is constructible and K has no real places, then $Ext^r_U(F, \mathbb{G}_m) = 0$ for $r > 4$.

Sketch. (a) Consider an open $V \subseteq U$ and use the exact sequence

$$H^r_c(V, F_V) \to H^r_c(U, F) \to \bigoplus_{v \in U \setminus V} H^r(K_v, F_v)$$

And for local Tate duality we have $H^r(K_v, F_v) = 0$ for $r > 3$, so we can consider F locally constant such that in the number field case $mF = 0$ for m invertible on U.
By definition, if we show that

$$H^r(U, F) \to \bigoplus_{v \text{ real}} H^r(K_v, F)$$

is an isomorphism, we have $H^r_c(U,F)=0$. We have that this morphism identifies with

$$H^r(G_S,F_\eta)\to\bigoplus_{v\text{ real}}H^r(K_v,F_\eta)$$

which is an isomorphism for $r\geq 3$ except if the order of F_η is divisible by $char(K)$, and using some technical lemmas one can show that in this case $H^r(U,F)=0$ for $r>3$ (see [Mil06, II.3.12]).
If now F is $\mathbb{Z}$-constructible, then F_{tor} is constructible and exact sequence

$$0\to F_{tor}\to F\to F_{tf}\to 0$$

shows that it is enough to show the theorem for F torsion free. Since we have the long exact sequence

$$0\to H^{r-1}_c(U,F/mF)\to H^r_c(U,F)\xrightarrow{m}H^r_c(U,F)$$

we have a surjection

$$H^{r-1}_c(U,F/mF)\twoheadrightarrow {}_mH^r_c(U,F)$$

and for the previous result $H^{r-1}_c(U,F/mF)=0$ for $r>4$, hence it is enough to show that $H^r_c(U,F)$ is torsion for $r>4$. But since we are assuming F locally constant, we have the long exact sequence

$$\bigoplus_{v\in S}H^{r-1}(K_v,F_v)\to H^r_c(U,F)\to H^r(U,F)$$

and since $\bigoplus_{v\in S}H^r(K_v,F_v)$ is finite for $r>0$, and $H^r(U,F)$ is torsion, we conclude

(b) Since K has no real places, for $r>3$ $H^r(U,F)=H^r_c(U,F)=0$. If F has support in a closed subset, i.e. it is of the form i_*F with $i:Z\to U$ a closed immersion, the result comes from the isomorphism

$$\mathrm{Ext}^r(i_*F,\mathbb{G}_m)\cong\mathrm{Ext}^{r-1}_Z(F,\mathbb{Z})$$

which is zero for $r>3$ for local Artin-Verdier duality. So we can assume F to be locally constant. So in this case $\mathcal{E}xt^r(F,\mathbb{G}_m)=0$ for $r>1$ and torsion for $r=0,1$, hence a direct limit of constructible sheaves (see lemma 2.2.3). So since $\mathbb{Z}$ is finitely presented, we have that if $\mathcal{E}xt^s(F,\mathbb{G}_m)=\varinjlim P_i$

$$H^r(U,\mathcal{E}xt^s(F,\mathbb{G}_m))=\varinjlim H^r(U,P_i)$$

which is zero for $r>3$, hence , since

$$\mathrm{Ext}^r_U(F,\mathbb{G}_m)=H^r(R\Gamma(U,R\mathcal{H}om(F,\mathbb{G}_m)))=\bigoplus_{i+j=r}H^i(U,\mathcal{E}xt^j(F,\mathbb{G}_m))$$

which is zero for $r>4$.

□

Lemma 4.3.8. *If $\widehat{\alpha}^r(X, \mathbb{Z}/m\mathbb{Z})$ is an iso for all r and $m \geq 0$ whenever K has no real places, then theorem 4.3.1 is true.*

Proof. The assumption says that theorem 4.3.1 is true for F constant on X, and so for lemma 4.3.5 implies that it is true for F constant on an open subset U.
For lemma 4.3.5, it is enough to prove the theorem on U such that F is locally constant on U and 2 is invertible on U in the number field case. The previous lemma says that if K has no real places $\widehat{\alpha}^r(U, F)$ is an isomorphism (it is the zero map) for $r < -1$, so we need the induction step. Consider $\pi : U' \to U$ an Ãľtale covering such that F is constant on U' and such that U' is the normalization of U on an extension K'/K with no real places. Then π_* is exact and $\pi_*\pi^*F \twoheadrightarrow F$ is epi, so consider the exact sequence

$$0 \to F' \to \pi_*\pi^*F \to F \to 0$$

Since π^*F is constant by definition, also $\pi_*\pi^*F$ is constant, so by hypothesis $\alpha^r(U, \pi_*\pi^*F)$ is an isomorphism. and F' is locally constant by definition. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Ext}_U^{r-1}(\pi_*\pi^*F, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_U^{r-1}(F', \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_U^{r}(F, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_U^{r}(\pi_*\pi^*F, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_U^{r}(F', \mathbb{G}_m) \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} & & \downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (4)} \\
H^{4-r}(U, \pi_*\pi^*F)^* & \longrightarrow & H^{4-r}(U, F')^* & \longrightarrow & H^{3-r}(U, F)^* & \longrightarrow & H^{3-r}(U, \pi_*\pi^*F)^* & \longrightarrow & H^{3-r}(U, F')^*
\end{array}$$

which can be replaced in degree 0 and 1 by the completion
So by induction hypothesis, (2) is an isomorphism, and by assumption (1) and (4) are isomorphism, so (2) is a mono for all locally constant sheaves F, then (4) is a mono, so (2) is an iso, hence the theorem is true. ☐

So from now on we will suppose K with no real primes, hence in this context $H_c^r(U, F) = H^r(X, j_!F)$, and in the case $U = X$ we have $H_c^r(X, F) = H^r(X, F)$.
We need now a technical result:

Lemma 4.3.9. *For any $\mathbb{Z}$-constructible sheaf F on U, there is a finite surjective map $\pi_1 : U_1 \to U$, a finite map $\pi_2 : U_2 \to U$ with finite image, constant $\mathbb{Z}$ constructible sheaves F_i on U_i , and an injective map $F \to \oplus \pi_{i*}F_i$.*

Proof. Let V be an open subset of U such that F_V is locally constant. Then there is a finite extension K' of K such that the normalization $\pi : V' \to V$ of V in K' is étale over V and $F_{V'}$ is constant. Let $\pi_1 : U_1 \to U$ be the normalization of U in K', and let F_1 be the constant sheaf on U_1 corresponding to the group $\Gamma(V', F_{V'})$. Then the canonical map $F_V \to \pi_*F_{V'}$ extends to a map $\alpha : F \to \pi_{1*}F_1$ whose kernel has support on $U \setminus V$. Now take U_2 to be an étale covering of $U \setminus V$ on which the inverse image of F on $V \setminus U$ becomes a constant sheaf, and take F_2 to be the direct image of this constant sheaf. ☐

We denote as usual the dual maps of $\alpha^{3-r}(U, F)$ as $\beta^r(U, F) : H_c^r(U, F) \to \mathrm{Ext}_U^{3-r}(F, \mathbb{G}_m)^*$, and we will first attack the theorem for constructible sheaves, where it is enough to prove that β^r is an isomorphism.

Lemma 4.3.10. *(a) Fix $r_0 > 0$. If for all $r < r_0$, all K and all F constructible on X β^r is an iso, then β^{r_0} is mono.*

(b) Moreover, assume that $\beta^{r_0}(X, \mathbb{Z}/n\mathbb{Z})$ is an iso if $\mu_n(K) = \mu_n(\overline{K})$, then $\alpha^{r_0}(X, F)$ is an iso for all K and all F constructible

Proof. (a) Consider a torsion flasque injection $F \to I$ (e.g. Godement resolution, which is torsion since F is constructible). So I is a filtered colimit of constructible sheaves, and since $H^{r_0}(X, I) = 0$ and filtered colimit is exact and commutes with the cohomology, for all $c \in H^{r_0}(X, F)$, $c \neq 0$ there exists an embedding $F \hookrightarrow F'$ with F' constructible and such that $c \mapsto 0$. Then, since $Q = F/F'$ is constructible, we have a morphism of long exact sequences:

$$\begin{tikzcd}
H^{r_0-1}(X,F') \arrow[r] \arrow[d, "(1)"] & H^{r_0-1}(X,Q) \arrow[r, "j_1"] \arrow[d, "(2)"] & H^{r_0}(X,F) \arrow[r, "i_1"] \arrow[d, "\beta^r"] & H^{r_0-1}(X,F') \arrow[r] \arrow[d] & \cdots \\
\mathrm{Ext}_U^{4-r_0}(F',\mathbb{G}_m) \arrow[r] & \mathrm{Ext}_U^{3-r_0}(F,\mathbb{G}_m) \arrow[r, "j_2"] & \mathrm{Ext}_U^{4-r_0}(F',\mathbb{G}_m) \arrow[r, "i_2"] & \mathrm{Ext}_U^{3-r_0}(F',\mathbb{G}_m) \arrow[r] & \cdots
\end{tikzcd}$$

Since $i_1(c) = 0$, then $c = j_1(c')$, and since $c \neq 0$ $c' \notin Im(H^{r_0-1}(X, F'))$ hence $\beta^r(c) = \beta^r(j_1(c')) = j_2(2)(c')$, and since (1) and (2) are isomorphisms, $(2)(c') \notin Im(\mathrm{Ext}_U^{4-r_0}(F', \mathbb{G}_m))$ hence $\beta^{r_0}(c) \neq 0$. This works for all c, so β^{r_0} is mono.

(b) Consider U small enough such that there exists a Galois extension K'/K with $\mu_m(K') = \mu_m(\overline{K})$ for some m such that $mF = 0$, and such that if U' is the normailzation of U in K', then $F_{U'}$ is locally constant and $U' \to U$ is étale. Following the construction of the previous lemma, we can take U_1 as the normalization of X in K' and consider $F \longrightarrow F_* := \pi_{1*}F_1 \oplus \pi_{2*}F_2$, then $\pi_{2*}F_2$ has support in a finite set, so $\beta^{r_0}(X, \pi_{2*}F_2)$ is an iso for lemma 4.3.3, and by hypothesis $\beta^{r_0}(U_1, F_1)$ is iso, so by lemma 4.3.5 $\beta^{r_0}(X, \pi_{1*}F_1)$ is, hence $\beta^{r_0}(X, F_*)$ is. So we have that again $Q := F_*/F$ is constructible, so we have a diagram:

$$\begin{tikzcd}
H^{r_0-1}(X,F) \arrow[r] \arrow[d, "(1)"] & H^{r_0-1}(X,Q) \arrow[r] \arrow[d, "(2)"] & H^{r_0}(X,F) \arrow[r] \arrow[d, "(3)"] & H^{r_0-1}(X,F') \arrow[r] \arrow[d, "(4)"] & \cdots \\
\mathrm{Ext}_U^{4-r_0}(F',\mathbb{G}_m) \arrow[r] & \mathrm{Ext}_U^{3-r_0}(F,\mathbb{G}_m) \arrow[r, "j_2"] & \mathrm{Ext}_U^{4-r_0}(F',\mathbb{G}_m) \arrow[r, "i_2"] & \mathrm{Ext}_U^{3-r_0}(F',\mathbb{G}_m) \arrow[r] & \cdots
\end{tikzcd}$$

Where (1), (2) and (4) are iso, so (3) is mono for all constructible sheaves, hence (5) is mono, so (3) is iso.

□

We can now attack theorem 4.3.1 in the constructible case:

Proof of theorem 4.3.1 in the constructible case. We prove by induction that β^r is an isomorphism:
For $r < 0$, it is the zero map, it follows from lemma 4.3.7.
Consider the long exact sequence

$$\mathrm{Ext}_X^r(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to H^r(X, \mathbb{G}_m) \xrightarrow{m} H^r(X, \mathbb{G}_m)$$

By the calculation on $H^r(X, \mathbb{G}_m)$, we have that $\mathrm{Ext}^3_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) = \frac{1}{m}\mathbb{Z}/\mathbb{Z}$, and since $H^0(X, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$, so $\beta^0(X, \mathbb{Z}/m\mathbb{Z})$ is an iso, hence $\beta^0(X, F)$ is an iso for the previous lemma, so $\beta^1(X, F)$ is always mono.
By class field theory, one can see that $\#H^1(X, \mathbb{Z}/m\mathbb{Z}) = \#Pic(X)_m = \#\mathrm{Ext}^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{G}_m)$, so $\beta^1(X, \mathbb{Z}/m\mathbb{Z})$ is a mono between finite groups of the same order, hence an iso, so $\beta^1(X, F)$ is an iso for all F, and $\beta^2(X, F)$ is always mono.
So it remains to show that for all $r \geq 2$ and all K such that $\mu_m(K) \cong \mu_m(\overline{K})$ we have $\beta^r(X, \mathbb{Z}/m\mathbb{Z})$ iso. So suppose now m prime with $char(K)$, hence $\mu_m(K) \cong \mu_m(\overline{K}) \cong \mathbb{Z}/n\mathbb{Z}$.
Consider $U \subseteq X$ where m is invertible and i the immersion of the complement, we have the morphism of exact sequences

$$\begin{tikzcd} H^r_c(U, \mathbb{Z}/m\mathbb{Z}) \arrow[r] \arrow[d, "\beta^r(U,\mathbb{Z}/m\mathbb{Z})"] & H^r(X, \mathbb{Z}/m\mathbb{Z}) \arrow[r] \arrow[d, "\beta^r(X,\mathbb{Z}/m\mathbb{Z})"] & H^r(X, i_*\mathbb{Z}/m\mathbb{Z}) \arrow[d, "\beta^r(X,i_*\mathbb{Z}/m\mathbb{Z})"] \\ \cdot & \cdot & \cdot \end{tikzcd}$$

So by five lemma $\beta^2(U, \mathbb{Z}/m\mathbb{Z})$ is mono. And since $\mathrm{Ext}^1_U(\mathbb{Z}/m\mathbb{Z}), \mathbb{G}_m \cong H^1(U, \mu_n)$ and $H^1_c(X, \mathbb{Z}/n\mathbb{Z})$ have the same number of elements (see [Mil06, II.2.13]), so $\beta^2(U, \mathbb{Z}/m\mathbb{Z})$ is an isomorphism.
β^3 comes from the pairing

$$\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{G}_m) \times H^3_c(U, \mathbb{Z}/m\mathbb{Z}) \to H^3(U, \mathbb{G}_m)$$

and since m is prime with $char(K)$, w have by hypothesis that there is a noncanonical isomorphism $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mu_n$, so since $H^3_c(U, \mathbb{Z}/m\mathbb{Z}) = \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ for Kummer since $H^2(U, \mathbb{G}_m) = 0$.
So β^3 is an iso, and since for $r > 3$ $H^r_c(X, \mathbb{Z}/m\mathbb{Z}) = 0$, $\beta^r = 0$ is an isomorphism, so $\beta^r(X, \mathbb{Z}/m\mathbb{Z})$ is always an isomorphism.
Let now $p = char(K) > 0$. We have Artin-Schreier

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$$

which gives $H^r(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r > 2$, so β^r is an iso for $r > 3$, and since $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m) = Ker(\mathrm{Hom}(\mathbb{Z}, \mathbb{G}_m) \xrightarrow{p} \mathrm{Hom}(\mathbb{Z}, \mathbb{G}_m)) = Ker(\mathbb{G}_m(X) \xrightarrow{p} \mathbb{G}_m(X))$, but X is chosen such that p is invertible, so $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m) = 0$ and β^3 is also an iso.
We need to show that β^2 is an iso, but we already know it is injective and by the same idea as before the groups have the same order, so β^2 is also an isomorphism. □

We can now prove it in full generality:

Proof of theorem 4.3.1. The only thing left to prove is that $\widehat{\alpha}^r(X, \mathbb{Z}) : H^r(X, \mathbb{G}_m)^\wedge \to H^{3-r}(X, \mathbb{Z})^*$ is an isomorphism. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$$

And since $\mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/n\mathbb{Z}$ for the previous theorem we have a canonical iso[4]

$$\varprojlim_n \mathrm{Ext}^r(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \cong H^{3-r}(X, \mathbb{Q}/\mathbb{Z})^*$$

[4] the cofiltered limit is exact for Mittag-Leffer conditions: the groups are finite

In particular, we have the exact sequence

$$H^r(X, \mathbb{G}_m) \xrightarrow{n} H^r(X, \mathbb{G}_m) \to \mathrm{Ext}^{r+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to {}_nH^r(X, \mathbb{G}_m) \to 0$$

So by Mittag-Leffer conditions (we have no real primes here) we can apply $\varprojlim_n$ and get

$$0 \to \varprojlim_n (nH^r(X, \mathbb{G}_m)) \to \varprojlim_n \mathrm{Ext}^{r+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \varprojlim_n ({}_nH^{r+1}(X, \mathbb{G}_m)) \to 0$$

Recall that if X is a number field with no real primes $\mathcal{O}_X(X)^\times$ is finitely generated, $Pic(X)$ is finite and $H^r(X, \mathbb{G}_m) = 0$ for $r \neq 2$, so $nH^r(X, \mathbb{G}_m)$ are cofinal between the open subgroups and so $\varprojlim_n (nH^r(X, \mathbb{G}_m)) = H^r(X, \mathbb{G}_m)^\wedge$. In the function field case $\mathcal{O}_X(X)^\times$ is finite, $Pic(X)$ is finitely generated and $H^r(X, \mathbb{G}_m) = 0$ for $r \neq 2$, so again $\varprojlim_n (nH^r(X, \mathbb{G}_m)) = H^r(X, \mathbb{G}_m)^\wedge$. Hence we have a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^r(X, \mathbb{G}_m)^\wedge \arrow[r] \arrow[d] & \varprojlim_n \mathrm{Ext}^{r+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \arrow[r] \arrow[d, "\sim"] & \varprojlim_n ({}_nH^{r+1}(X, \mathbb{G}_m)) \arrow[r] & 0 \\
& H^{3-r}(X, \mathbb{Z})^* \arrow[r] & H^{2-r}(X, \mathbb{Q}/\mathbb{Z})^* \arrow[r] & H^{2-r}(X, \mathbb{Q})^* &
\end{tikzcd}$$

And since $R\Gamma(X, \mathbb{Q}) = R\mathrm{Hom}_X(\mathbb{Z}, \mathbb{Q}) = R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Q})$, so

$$H^{2-r}(X, \mathbb{Q}) = H^{3-r}(X, \mathbb{Q}) = 0 \text{ for } 2 - r > 0$$

Hence for $r \leq 1$ we have $H^{2-r}(X, \mathbb{Q}/\mathbb{Z}) \cong H^{3-r}(X, \mathbb{Z})$, and for $r \leq 1$, $H^{r+1}(X, \mathbb{G}_m)$ is finitely generated, so $\varprojlim_n ({}_nH^{r+1}(X, \mathbb{G}_m)) = 0$, so $H^r(X, \mathbb{G}_m)^\wedge \cong \varprojlim_n \mathrm{Ext}^{r+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$, and so $\widehat{\alpha}^r(X, \mathbb{Z})$ is an iso for $r \leq 1$.
For $r > 3$ and $r = 2$ it is the zero map, so the only one left to see is $r = 3$, which is

$$H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} \to H^0(X, \mathbb{Z})^* = \mathbb{Z}^*$$

which is obviously an isomorphism. □

Remark 4.3.11. In the context of derived category, if $F \in D^+_{cons}(U)$, then we have quasi isomorphisms

$$\mathrm{Hom}_{D(X)}(F, \mathbb{G}_m[3 - r]) \cong \mathrm{Hom}_{\mathbb{Z}}(R\Gamma_c(X, F[r]), \mathbb{Q}/\mathbb{Z})$$

Corollary 4.3.12. *Let m be invertible on U and F be a constructible sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules. Then we have a perfect pairing of $\mathbb{Z}/m\mathbb{Z}$-modules:*

$$H^r_c(U, F) \times Ext^{3-r}_{Sh(U,\mathbb{Z}/m\mathbb{Z})}(F, \mu_m) \to \mathbb{Z}/m\mathbb{Z}$$

Proof. Since $\mathrm{Ext}^{3-r}_{Sh(U,\mathbb{Z}/m\mathbb{Z})}(F, \mu_m) \cong \mathrm{Ext}^{3-r}_U(F, \mathbb{G}_m)$ we have by Artin-Verdier duality a perfect pairing

$$H^r_c(U, F) \times \mathrm{Ext}^{3-r}_{Sh(U,\mathbb{Z}/m\mathbb{Z})}(F, \mu_m) \to H^3_c(U, \mu_m) = \mathbb{Z}/m\mathbb{Z}$$

The last equality follows form Kummmer. □

4.3.1 An application

Let K be a number field, and let $F = \mathbb{Z}$, so $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{G}_m) = H^1(X, \mathbb{G}_m) = Cl(X)$ is finite. On the other hand, since $H^r(X, \mathbb{Q}) = 0$ for $r > 1$ and $\widehat{H}^r(G_{\mathbb{R}}, \mathbb{Q}) = 0$ for all r, so $H^r_c(X, \mathbb{Q}) = 0$ for $r > 1$. Hence $H^2_c(X, \mathbb{Z}) = H^1_c(X, \mathbb{Q}/\mathbb{Z})$.
Since $H^1(G_{\mathbb{R}}, \mathbb{Q}/\mathbb{Z})^* = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z})^* = \mathbb{Z}/2\mathbb{Z}$, if a is the number of real embeddings, we have the exact sequence

$$\left(\mathbb{Z}/2\mathbb{Z}\right)^a \to H^1(X, \mathbb{Q}/\mathbb{Z})^* = \pi_1(X)^{ab} \to H^1_c(X, \mathbb{Q}/\mathbb{Z})^* \to 0$$

So Artin-Verdier duality gives an isomorphism

$$Cl(X) \xrightarrow{\sim} \pi_1(X)^{ab}/M$$

Where M is a 2-primary component. In particular we find again the Hilbert Class Field: $\pi_1(X)^{ab}$ classifies all the unramified extensions of K and M individuates the extensions which ramify at infinity. So $\pi_1(X)^{ab}/M \cong Gal(H_K/K)$ and the isomorphism is the classical one

$$\begin{aligned} Cl(X) &\to \pi_1(X)^{ab}/M \\ \wp &\mapsto Fr_{\wp} \end{aligned}$$

Chapter 5

Higher dimensions

5.1 Statement of the duality theorem

Let us keep the notation of the previous chapters. Now $\pi : Y \to U$ will be a separated morphism of finite type pure of dimension d, and for any $F \in D^b_{cft}(Y)$ we will define

$$R\Gamma_c(X, F) = R\Gamma_c(U, R\pi_! F)$$

Remark that as seen in theorem D.3.4, if $F \in D^b_{ctf}(Y)$ then $R\pi_! F \in D^b_{ctf}(U)$, so as it is seen in Artin-Verdier proof, $H^r_c(Y, R\pi_! F)$ are finite.
Recall that the trace map defined in Section 3.1 gives an isomorphism $R^{2d}\pi_!\mu_m^{\otimes d} \cong \mathbb{Z}/m\mathbb{Z}$, hence tensoring with μ_m we have an isomorphism

$$R^{2d}\pi_!\mu_m^{\otimes d+1} \cong \mu_m$$

We have now that μ_m is a flat sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules, hence $\mu_m \otimes^{\mathbb{L}} \mu_m = \mu_m \otimes \mu_m$, so we can use the definition

$$\mathbb{Z}/m\mathbb{Z}(d) := \mu_m^{\otimes d}$$

also on the derived category. We have that

$$H^{2d+3}_c(Y, \mathbb{Z}/m\mathbb{Z}(d+1)) \cong \mathbb{H}^{2d+3}(R\Gamma_c(U, R\pi_!\mathbb{Z}/m\mathbb{Z}(d+1)) = \bigoplus_{r=0}^{2d+3} H^r_c(U, R^{2d+3-r}\pi_!\mathbb{Z}/m\mathbb{Z}(d+1))$$

And since $R^r\pi_!\mathbb{Z}/m\mathbb{Z}(d+1)$ is constructible on U, we have Artin-Verdier duality, so

$$H^r_c(U, R^{2d+3-r}\pi_!\mathbb{Z}/m\mathbb{Z}(d+1)) \cong \operatorname{Ext}^{3-r}(R^{2d+3-r}\pi_!\mathbb{Z}/m\mathbb{Z}(d+1), \mathbb{Z}/m\mathbb{Z}(1))^*$$

Hence it is zero for $r \neq 0, 1, 2, 3$, but since $R^r\pi_!\mu_m^{\otimes d+1} = 0$ for $r > 2d$, we have

$$\mathbb{H}^{2d+3}_c(R\Gamma_c(U, R\pi_!\mathbb{Z}/m\mathbb{Z}(d+1)) = H^3_c(U, R^{2d}\pi_!\mathbb{Z}/m\mathbb{Z}(d+1)) \cong H^3_c(U, \mu_m)$$

and by Kummer theory and the fact that $H^2(U, \mathbb{G}_m) = 0$, we have $H^3_c(U, \mu_m) = \mathbb{Z}/m\mathbb{Z}$ we have a trace map

$$H^{2d+3}_c(Y, \mathbb{Z}/m\mathbb{Z}(d+1)) \cong \mathbb{Z}/m\mathbb{Z}$$

and a pairing

$$\operatorname{Ext}^r_{Sh(Y,m)}(F, \mathbb{Z}/m\mathbb{Z}(d+1)) \times H^{2d+3-r}_c(Y, F) \to H^{2d+3}_c(Y, \mathbb{Z}/m\mathbb{Z}(d+1)) \cong \mathbb{Z}/m\mathbb{Z} \quad \text{(Pairing 5.1)}$$

Theorem 5.1.1. *Let π be smooth separated of pure dimension d, F constructible such that $mF = 0$. Then Pairing 5.1 is perfect.*

Proof. We have that according to [AGV72, XVIII] $R\pi^!\mu_m \cong \mu_m^{\otimes d+1}[2d]$ and

$$R\pi_* R\mathcal{H}om_{Sh(Y,\mathbb{Z}/m\mathbb{Z})}(F, R\pi^!\mu_m) \cong R\mathcal{H}om_{Sh(U,\mathbb{Z}/m\mathbb{Z})}(R\pi_! F, \mu_m)$$

So, on the LHS we have

$$R\Gamma(U, R\pi_* R\mathcal{H}om_{Sh(Y,\mathbb{Z}/m\mathbb{Z})}(F, \mathbb{Z}/m\mathbb{Z}(d+1)[2d])) = R\Gamma(Y, R\mathcal{H}om_{Sh(Y,\mathbb{Z}/m\mathbb{Z})}(F, R\pi^!\mathbb{Z}/m\mathbb{Z}(d+1))[2d])$$
$$R\mathrm{Hom}_{D(Y,\mathbb{Z}/m\mathbb{Z})}(F, \mathbb{Z}/m\mathbb{Z}(d+1))[2d])$$

And on the RHS we have

$$R\Gamma(U, R\mathcal{H}om_{Sh(U,\mathbb{Z}/m\mathbb{Z})}(R\pi_! F, \mu_m) = \mathrm{Hom}_{D(Y,\mathbb{Z}/m\mathbb{Z})}(R\pi_! F, \mu_m) = \mathrm{Hom}_{D(Y)}(R\pi_! F, \mathbb{G}_m)$$

So we have by Artin-Verdier duality

$$\mathrm{Ext}^{2d+r}_{Sh(Y,\mathbb{Z}/m\mathbb{Z})}(F, \mathbb{Z}/m\mathbb{Z}(d+1)) = \mathrm{Ext}^r_U(R\pi_! F, \mathbb{G}_m) \cong H_c^{3-r}(U, R\pi_! F)^*$$

And this proves the theorem □

Recall that if $i > 0$ and $mF = 0$ we defined $F(i) = F \otimes \mathbb{Z}/m\mathbb{Z}(i)$ and $F(-i) = \mathcal{H}om(F, \mu_m^{\otimes i})$. By the same argument as before and by lemma 3.2.2, it passes to the derived category

Corollary 5.1.2. *In the hypotheses above, if F is locally constant we have a perfect pairing*

$$H^r(Y, F(-d-1)) \times H_c^{2d+3-r}(Y, F) \to \mathbb{Z}/m\mathbb{Z}$$

Proof. Since

$$R\Gamma(Y, R\mathcal{H}om(F, \mathbb{Z}/m\mathbb{Z}(d+1))) \cong R\mathrm{Hom}(F, \mathbb{Z}/m\mathbb{Z}(d+1))$$

we have that $H^r(Y, F(-d-1)) = \mathrm{Ext}^r(F, \mathbb{Z}/m\mathbb{Z}(d+1))$, so it follows from Pairing 5.1 □

5.2 Motivic cohomology

5.2.1 Locally logarithmic differentials

Let S be a perfect scheme of characteristic p and $X \to S$ be an S-scheme. As it is done in Section D.1, we have the Frobenius map

$$Fr : X \to X$$

So $Fr_* : Sh_{\tilde{A}rt}(X, \mathcal{O}_X) \to Sh_{\tilde{A}rt}(X, \mathcal{O}_X)$ acts as the identity on the abelian group and changex the action of $\mathcal{O}_X$ by $f\alpha \mapsto f^p\alpha$. In particular it preserves locally free $\mathcal{O}_X$-modules.
Recall, as it is defined in [Sta, Tag 01UM] the sheaf of differential forms $\Omega^1_{Y/S}$ and consider the complex

$$0 \to \mathcal{O}_X \xrightarrow{d^0} \Omega^1_{Y/S} \dots \Omega^r_{Y/S} = \wedge^r_{\mathcal{O}_Y} \Omega^1_{Y/S} \xrightarrow{d^r} \dots$$

Let $\Omega^r_{Y/S,cl}$ be the kernel of d^r, i.e. the sheaf of closed r-forms.

Lemma 5.2.1. *We can define a unique family of maps* $C^r : \Omega^r_{Y/S,cl} \to \Omega^r_{Y/S}$ *such that*

(a) $C^r(1) = 1$

(b) $C^r(f^p\omega) = fC^r(\omega)$ *for all* $f \in \mathcal{O}_X$

(c) $C^{r+s}(\omega \wedge \omega') = C^r(\omega) \wedge C^r(\omega')$

(d) $C^r(\omega) = 0$ *if and only if* $\omega = d^{r-1}(\omega')$

(e) $C^1(f^{p-1}d^0 f) = df$.

Proof. [Mil76]

Remark 5.2.2. Since $\Omega^0_{Y/S,cl} = (\mathcal{O}_X)^p \cong \mathcal{O}_X$, for all $f^p \in \Omega^0_{Y/S,cl}$ we have that $C(f^p) = f$, so in degree 0 the Cartier map is the Frobenius.

Theorem 5.2.3. *The map* $C^r - id$ *is epi, so if we denote its kernel as* $\nu_1(r)$ *we have an exact sequence*

$$0 \to \nu_1(n) \to \Omega^r_{Y/S,cl} \xrightarrow{id-C} \Omega^r_{Y/S} \to 0$$

Proof. Consider a geometric point P and a suitably small neighborhood U such that we have a local system $x_1 \dots x_m$ and let $u_i = x_i - 1$ Choosing U suitably small we have u_i invertible.
So for every $\omega \in \Omega^r_{Y/S}(U)$ we have that there are $f_j \in \mathcal{O}_X(U)$ such that we can write

$$\omega = \sum f_j \frac{du_{j_1}}{u_{j_1}} \wedge \dots \wedge \frac{du_{j_r}}{u_{j_r}}$$

As now, by definition of C^r we have

$$C^1(\frac{du}{u}) = C^1((\frac{1}{u})^p u^{p-1} du) = \frac{1}{u} C^1((u^{p-1}du) = \frac{du}{u}$$

So we have:

$$(id - C^r)(g^p \frac{du_{j_1}}{u_{j_1}} \wedge \dots \wedge \frac{du_{j_r}}{u_{j_r}}) = (g^p - g)(\frac{du_{j_1}}{u_{j_1}} \wedge \dots \wedge \frac{du_{j_r}}{u_{j_r}})$$

In other words, we need to prove that there exists an Ãľtale neighbourhood of P such that there is g such that $g^p - g = f_j$. This can be done by taking the Artin-Schreier unramified extension of the Zariski local ring $\mathcal{O}_{X,P}$. □

With the same idea, one can define $\nu_n(r)$ as the kernel of the map induced to the n-Witt vectors $W_n(\Omega^r_{Y/S,cl}) \to W_n(\Omega^r_{Y/S})$, which is an exact functor.
The wedge product pairing on $\Omega^\bullet_{Y/S}$ defines a cup product pairing

$$\nu_n(i) \times \nu_n(j) \to \nu_n(i + j)$$

Theorem 5.2.4. *Let* Y *be a smooth proper variety of dimension* d *over a finite field* k. *Then we have a trace isomorphism* $H^{d+1}(Y, \nu_n(d)) \cong \mathbb{Z}/p^n\mathbb{Z}$ *and the cup product induces a perfect pairing of finite groups*

$$H^r(Y, \nu_n(i)) \times H^{d+1-r}(Y, \nu_n(d - i)) \to \mathbb{Z}/p^n\mathbb{Z}$$

Proof. See [Mil76] for the case $n = 1$ or $dim(Y) \leq 2$, [Mil86] for the general case.

5.2.2 Motivic cohomology

Let Y be a regular scheme over a field of characteristic p (can also be zero). Lichtenbaum conjectured the existence of objects $\mathbb{Z}(r) \in D(Y_{et})$ such that

(a) $\mathbb{Z}(0) = \mathbb{Z}$, $\mathbb{Z}(1) = \mathbb{G}_m[-1]$

(b) For $\ell \neq p$ and for all n there is a triangle

$$\mathbb{Z}(i) \xrightarrow{\ell^n} \mathbb{Z}(i) \to \mathbb{Z}/\ell^n\mathbb{Z}(i) \tag{b_ℓ 5.2}$$

and there is a triangle

$$\mathbb{Z}(i) \xrightarrow{p^n} \mathbb{Z}(i) \to \nu_n(i)[-i] \tag{b_p 5.3}$$

(c) We have canonical pairings $\mathbb{Z}(i) \times \mathbb{Z}(j) \to \mathbb{Z}(i+j)$

(d) $H^{2r-j}(\mathbb{Z}(i)) = Gr^r_\gamma(\mathcal{K}_j)$ (the γ-filtration of Quillen K-sheaves) up to small torsion, $H^r(\mathbb{Z}(i)) = 0$ for $r > i$ and $r < 0$. If $i \neq 0$, also $H^0(\mathbb{Z}(i)) = 0$.

(e) If Y is a smooth complete variety over a finite field, then $H^(Y, \mathbb{Z}(i)$ is torsion for all $r \neq 2i$, and $H^{2i}(Y, \mathbb{Z}(i))$ is finitely generated.

(f) (Purity) If Y is smooth and $i : Z \to Y$ is a closed immersion of relative dimension c, then if $j > c$ $Ri^!\mathbb{Z}(j) = \mathbb{Z}(j-c)[-2c]$

There is a candidate for these object proposed by Bloch in [Blo86].

Theorem 5.2.5. *Let $\pi : Y \to U$ be smooth proper pure of dimension d. Let ℓ be a prime such that either ℓ is invertible on U or $\ell = char(K)$. Assume that there exist complexes $\mathbb{Z}(i)$ satisfying Equation (b_ℓ 5.2) and that $H_c^{2d+3}(Y, \mathbb{Z}(d+1))$ is torsion. Then we have a canonical isomorphism*

$$H_c^{2d+4}(Y, \mathbb{Z}(d+1))(\ell) \cong (\mathbb{Q}/\mathbb{Z})(\ell)$$

and the pairing

$$H^r(Y, \mathbb{Z}(i))(\ell) \times H_c^{2d+4-r}(Y, \mathbb{Z}(d+1-i))(\ell) \to H^{2d+4}(Y, \mathbb{Z}(d+1))(\ell) \cong (\mathbb{Q}/\mathbb{Z})(\ell)$$

kills only the divisible subgroups.

Proof. If $\ell \neq char(K)$, the triangle of Equation (b_ℓ 5.2) gives for all n a long exact sequence

$$\xrightarrow{\ell^n} H_c^{2d+3}(Y, \mathbb{Z}(d+1)) \to H_c^{2d+3}(Y, \mu_{\ell^n}^{\otimes d+1}) \cong \mathbb{Z}/\ell^n\mathbb{Z} \to H_c^{2d+4}(Y, \mathbb{Z}(d+1)) \xrightarrow{\ell^n}$$

So by taking the ℓ-torsion and passing to the limit we have an isomorphism

$$\varinjlim \mathbb{Z}/\ell^n\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})(\ell) \xrightarrow{\sim} H_c^{2d+4}(Y, \mathbb{Z}(d+1))(\ell)$$

The second statement follows from the long exact sequences for $H^\bullet$ and $H_c^\bullet$. The proof for $\ell = p$ is similar considering the triangle Equation (b_p 5.3). $\square$

Appendix A

Global class field theory

A.1 AdÃĺle and IdÃĺle

Throughout this section, k would be a global field, i.e. a number field or a finite separable extension of $\mathbb{F}_p(T)$, the places would be normalized absolute values, $S_{k\infty} = S_r \cup S_c$ would be the set of archimedean places with S_r real and S_c complex, and $S_k = S_{k\infty} \cup S_{kf}$ would be the set of all places.
If k is a number field, $Div(\mathcal{O}_k)$ is the group of fractional ideal and Cl_k is the class group. If k is a function field, then fixing X the corresponding integral proper smooth curve, $Div(X)$ is the group of divisors, $Div^0(X)$ is the kernel of $deg : Div(X) \to \mathbb{Z}$ and $Pic^0(X) = Div^0(X)/k^*$ with the diagonal embedding of k^*.
For uniformizing the notation, the group of divisors will be denoted multiplicatively.
I recall two basic but important results:

Theorem A.1.1 (Ostrowski). *1. If $k = \mathbb{Q}$, then*

$$S_{\mathbb{Q}} = \{|\cdot|_\infty, |\cdot|_p\}$$

Where $|\cdot|_\infty$ is the usual archimedean absolute value and $|\cdot|_p$ is the absolute value induced by the p-adic valuation for every prime number $p \in \mathbb{Z}$

2. If $k = \mathbb{F}_p[T]$, then

$$S_{\mathbb{Q}} = \{|\cdot|_\infty, |\cdot|_f\}$$

where $|\cdot|_\infty$ is induced by the degree valuation and $|\cdot|_f$ is induced by the f-adic valuation for every irreducible polynomial $f \in \mathbb{F}_p[T]$

Proof. [Neu13, II.3.7]

Corollary A.1.2 (Product formula). *If $k = \mathbb{Q}$ or $\mathbb{F}_p(T)$, then for all $\alpha \in k$ we have*

$$\prod_{v \in S} |\alpha|_v = 1$$

Theorem A.1.3 (Extension of valuations). *If $L = k[a]$ is a separable extension with $[L : k] = n$ and $v \in S_k$, then there are at most n extensions of v corresponding to the irreducible factors of the polynomial f_a in k_v*

Proof. [Neu13]

It is worth it to recall

Theorem A.1.4. *If K is complete with respect to an archimedean absolute value, then $k \cong \mathbb{R}$ or $k \cong \mathbb{C}$*

Hence for any local field the set of its places is well determined.

Theorem A.1.5 (Weak approximation theorem)**.** *If $v_1 \cdots v_m \in S_k$ are distinct places and $\alpha_1 \cdots \alpha_m \in k$, then for every $\epsilon > 0$ there is $\alpha \in k$ such that $|\alpha - \alpha_m|_{v_m} > \epsilon$*

Proof. [SD01, Theorem 17]

Lemma A.1.6. *If $\alpha \neq 0$ in k there are only finitely many places v such that $|\alpha|_v > 1$.*

Proof. [CF67, II.12]

With this results, we see that for any $\alpha \in k$, then $\alpha \in \mathcal{O}_v$ for almost all $v \in S_k$ Then we can define using the notion of restricted topological product ([CF67, II.13])

Definition A.1.7 (The ring of adÃĺles)**.** $\mathbb{A}_k = \prod_v^{\mathcal{O}_v} k_v$. With this definition $\mathbb{A}_k$ is locally compact and there is a natural inclusion $k \hookrightarrow \mathbb{A}_k$ given by the diagonal. The elements in the image of this map are called the *principal adÃĺles* and they will be still called k

Lemma A.1.8 (Product formula)**.** *If L/k is a separable extension, there is a topological isomorphism*

$$\mathbb{A}_k \otimes_k L \cong \mathbb{A}_L$$

which maps $k \otimes L \to L$

Proof. Conside $\omega_1 \cdots \omega_n$ a basis. The LHS is just

$$\prod_{v \in S_k}^{\oplus \omega_i \mathcal{O}_v} \oplus \omega_i k_v$$

Which by the extension theorem is topologically isomorphic to

$$\prod_{v \in S_k, V|v}^{\mathcal{O}_{L,V}} L_V$$

□

Lemma A.1.9. *$\mathbb{A}_k/k$ is compact and k is discrete*

Proof. For the prevous lemma, it is enough to prove it for $k = \mathbb{Q}$ or $k = \mathbb{F}_p(T)$. The weak approximation theorem says that for every adÃĺle $(\alpha_v)_v$ there exists a principal adÃĺle α such that $\alpha_v - \alpha \in \mathcal{O}_v$, i.e. every coset of k meets $\prod_{S_\infty} k_v \times \prod'_{S_f} \mathcal{O}_v$. Since $\prod_{S_\infty} k_v/\mathcal{O}_k$ is compact, there is a compact subset T of $\prod_{S_\infty} k_v$ that meets every coset of k, i.e. $T \times \prod'_{S_f} \mathcal{O}_v \to \mathbb{A}_k/k$

is surjective and continuous, hence $\mathbb{A}_k/k$ is compact since $T \times \prod'_{S_f} \mathcal{O}_v$ is.
k is trivially discrete since

$$D = \{(\alpha_v)_v : |\alpha_v|_v < 1 \text{ if } v \in S_\infty \& \ \alpha_v \in \mathcal{O}_v \text{ if } v \in S_\infty\}$$

is an open subset of $\mathbb{A}_k$ and $D \cap k = \{0\}$ for the product formula. □

Since $\mathbb{A}_k$ is locally compact, it admits a unique normalized Haar measure μ (see [Rud87]), and since $\mathbb{A}_k/k$ is compact, it has finite measure. We normalize the Haar measure such that $\mathbb{A}_k/k$ has measure 1.

Corollary A.1.10 (Product formula). $\prod |\alpha|_v = 1$ *for every nonzero principal adÃĺle.*

Proof. if $\alpha \in k$ then

$$\mu(\alpha X) = \prod |\alpha|_v \mu(X)$$

but since $\mathbb{A}/k$ has measure 1, this means $\prod |\alpha|_v = 1$ □

Definition A.1.11 (The group of idÃĺles). $\mathbb{I}_k = \prod_v^{\mathcal{O}_v^\times} k_v^\times$. With this definition $\mathbb{I}_k$ is group-theoretically isomorphic to $\mathbb{A}_k^\times$, but the topology is strictly finer (since $(\_)^{-1}$ is not continuous on $\mathbb{A}_k$), so there is a continuous inclusion $\mathbb{I}_k \hookrightarrow \mathbb{A}_k$ and a continuous multiplication $\mathbb{I}_k \times \mathbb{A}_k \to \mathbb{A}_k$. There is also the diagonal inclusion $k^\times \to \mathbb{I}_k$ and the elements in the image of this map are called the *principal idÃĺles* and they will be still called $k^\times$

Remark A.1.12. The topology of $\mathbb{I}_k$ is finer then the topology induced by $\mathbb{A}_k$ and $k^\times$ is already discrete in $\mathbb{A}_k$, so $k^\times$ is discrete in $\mathbb{I}_k$.

Definition A.1.13. We have a continuous map $c : \mathbb{I}_k \to \mathbb{R}^{>0}$ given by $(\alpha_v)_v \mapsto \prod |\alpha_v|_v$. We define the 1-idÃĺle $\mathbb{I}_k^0$ as the kernel of c. Notice that by the product formula $k^\times \subseteq \mathbb{I}_k^0$

Remark A.1.14. If k is a number field, then c is surjective: it is enough to take an idÃĺle which has 1 at every non-archimedean places and all archimedean places but one, so it follows from the surjectivity of the archimedean absolute value.

Lemma A.1.15. $\mathbb{I}_k^0$ *is closed in* $\mathbb{A}_k$*, hence it is closed in* $\mathbb{I}_k$ *and the two topologies coincide.*

Proof. [CF67, II.16]

Lemma A.1.16. *There is a constant C depending only on k such that for every $\alpha \in \mathbb{A}_k$ such that $\prod |\alpha_v|_v > C$ there is $\eta \in k^\times$ such that for all v*

$$|\eta|_v \leq |\alpha_v|_v$$

Proof. [CF67, II.13]

Theorem A.1.17. $\mathbb{I}_k^0/k^\times$ *is compact*

Proof. It is enough to find a compact $W \subseteq \mathbb{A}_k$ such that $W \cap \mathbb{I}_k^0 \to \mathbb{I}_k^0/k^\times$ is surjective. Take C as in lemma A.1.16 and α such that $c(\alpha) > C$, then

$$W := \{\xi : |\xi_v|_v \leq |\alpha_v|_v\}$$

Consider $\beta \in \mathbb{I}_k^0$, then $c(\beta^{-1}\alpha) = c(\alpha) > C$, hence by lemma A.1.16 there exists η such that

$$|\eta|_v \leq |\beta_v^{-1}\alpha_v|_v$$

Hence $\eta\beta \in W \cap \mathbb{I}_k^0$, so $W \cap \mathbb{I}_k^0 \to \mathbb{I}_k^0/k^\times$ is surjective. □

Corollary A.1.18 (Class group). *If k is a number field, the class group $Div(\mathcal{O}_k)/k^\times$ of k is finite. If k is a function field of a curve X over a finite field, $Pic^0(X)/k^*$ is finite.*

Proof. If $D = Div(\mathcal{O}_k)$ or $D = Div(X)$ is taken with the discrete topology, there is a continuous map

$$\beta \mapsto \prod_{v \in S_f} \wp_v^{v(\beta)} : \mathbb{I}_k^0 \to D$$

and by definition the image of k^* gives the principal divisors, hence $Im(\mathbb{I}_k^0)/k^\times$ is compact and discrete, so finite.
If k is a number field, $Im(\mathbb{I}_k^0) = Div(\mathcal{O}_k)$ since if $I = \prod \wp_i^{n_i}$, consider π a uniformizer of π_i and the idÃĺle η given by $\pi_i^{n_i}$ in the places $\wp_i$, in one archimedean place put $\frac{1}{\prod |\pi_i|_{\wp_i}}$ and 1 in all the other places, so $\eta \in \mathbb{I}_k^0$ and $\eta \mapsto I$
If k is a function field, this fails and for the surjectivity we need to restrict to $Div^0(X)$ since the non-archimedean valuaton $||_\infty$ is not surjective and does not allow us to compensate.

Corollary A.1.19 (Unity). *If S is finite and contains all the archimedean places, then $H_S = \{\eta \in k^\times : |\eta|_v = 1, v \notin S\}$ is the direct sum of a finite cyclic group of roots of 1 and a free abelian group of order $\#S - 1$.*

Proof. This description is given by the map

$$\eta \mapsto (\log(|\eta|_v)) : H_s \to \prod_S \mathbb{R}$$

it has kernel $\mu(k)$ and image a complete lattice in the hyperplane $\sum_{v \in S} x_v = 0$. See [Neu13, VI.1.1]

A.2 The IdÃĺle class group

Definition A.2.1. The *idÃĺle class group* is the Hausdorff locally compact group

$$C_k := \mathbb{I}_k/k^\times$$

Definition A.2.2. A *modulus* is a formal product

$$\mathfrak{M} = \prod_S \wp_v^{n_v}$$

where $n_v = 0$ if v is complex, $n_v = 0$ or 1 if v is real and $n_v = 0$ for almost all v. If $v \in S_f$, then take $U_v^0 = \mathcal{O}_v^\times$ and in the other cases:

$$U_v^{(n_v)} := \begin{cases} 1 + \wp_v^{n_v} \subseteq K_v^\times & \text{if } v \in S_f \\ \mathbb{R}^\times & \text{if } v \in S_r \text{and } n_v = 0 \\ \mathbb{R}_{>0} & \text{if } v \in S_r \text{and } n_v = 1 \\ \mathbb{C}^\times & \text{if } K_v = \mathbb{C} \end{cases}$$

Hence $x_v \in U_v^{n_v}$ means $x_v \in 1 + \wp_v^{n_v}$ if v is finite, $x_v > 0$ if v is real, nothing if v is complex

Definition A.2.3. Consider a modulus $\mathfrak{M}$, and take the open subgroup

$$\mathbb{I}_k^{\mathfrak{M}} := \prod_{\mathfrak{M}} U_v^{(n_v)}$$

Then we have the *congruence subgroup* mod $\mathfrak{M}$ given by

$$\mathbb{I}_k^{\mathfrak{M}} k^*/k^* \subseteq C_k$$

and the *ray class group*

$$C_k/C_k^{\mathfrak{M}}$$

Proposition A.2.4. *A subgroup of C_k is closed of finite index if and only if it contains $C_k^{\mathfrak{M}}$ for some $\mathfrak{M}$.*

Proof. $C_k^{\mathfrak{M}}$ is open and it is contained in $\mathbb{I}_k^{S_\infty} = \prod_{S_\infty} K_v^\times \times \prod_{S_f} U_v^\times$. Consider the map

$$(\alpha) \mapsto \prod \wp_v^{v(\alpha_v)} : C_k \to Cl_k \text{ or } Pic^0(X)$$

it is surjective and has kernel $\mathbb{I}_k^{S_\infty} k^\times/k^\times$.
$[C_k : \mathbb{I}_k^{S_\infty} k^\times/k^\times] = h$ where $h = \#Cl_k$ or $\#Pic^0(X)$. So

$$[C_k : C_k^{\mathfrak{M}}] = h[\mathbb{I}_k^{S_\infty} k^\times/k^\times : C_k^{\mathfrak{M}}] = h2^r \prod_{S_f} [U_v : U_v^{(n_v)}] < \infty$$

where r is the number of real places. Hence $C_k^{\mathfrak{M}}$ is open of finite index, so closed, and every subgroups that contains $C_k^{\mathfrak{M}}$ is the union of finitely many cosets of $C_k^{\mathfrak{M}}$, so it is closed of finite index.
Conversely, take N closed of finite index, it is open, so its preimage in $\mathbb{I}_k$ contains a neighborhood of 1 of the form

$$W = \prod_{S_\infty} W_v \times \prod_{S_f} 1 + \wp_v^{n_v}$$

where W_v is an open ball centered in 1 and n_v are suitable integers. If v is real, we can choose W_v small enough such that $W_w \subseteq \mathbb{R}_{>0}$. So the subgroup of $\mathbb{I}_k$ generated by W is of the form $\mathbb{I}_k^{\mathfrak{M}}$ with $\mathfrak{M} = \prod \wp_v^{n_v}$, hence $C_k^{\mathfrak{M}} \subseteq N$. □

Another definition of the ray class group:

Proposition A.2.5. *Let $J_k^{\mathfrak{M}} \subseteq Div(\mathcal{O}_k)$ (or $Div^0(X)$) be the group of divisors prime to $\mathfrak{M}$, and let $P_k^{\mathfrak{M}} \subseteq k^*$ be the subgroup (a) such that if v is finite, $a = 1 \bmod \mathfrak{M}$, and if v is real, $\sigma_v(a) > 0$ where $\sigma_v : k \to \mathbb{R}$ is the corresponding embedding. There is an isomorphism*

$$C_k/C_k^{\mathfrak{M}} \cong J_k^{\mathfrak{M}}/P_k^{\mathfrak{M}} =: Cl_k^{\mathfrak{M}}$$

Proof. Consider

$$\mathbb{I}_k^{\mathfrak{M}} := \{\alpha \in \mathbb{I}_k : \alpha \in U_v^{n_v}\}$$

Then $\mathbb{I}_k = \mathbb{I}_k^{\mathfrak{M}} k^*$ since by the approximation theorem for every $\alpha \in \mathbb{I}_k$ there is an $x \in k^*$ such that $\alpha_v x = 1 \bmod \wp^{n_v}$ for v finite and $\alpha_v x > 0$ for v real, so $\beta := \alpha x \in \mathbb{I}_k^{\mathfrak{M}}$ and $\alpha = \beta x^{-1}$.
If $a \in \mathbb{I}_k^{\mathfrak{M}} \cap k^*$, then by definition $a \in P_k^{\mathfrak{M}}$, so there is a surjective map

$$\alpha \mapsto (\alpha) = \prod_{S_f} \wp_v^{v(\alpha_v)} : C_k = \mathbb{I}_k^{\mathfrak{M}}/(\mathbb{I}_k^{\mathfrak{M}} \cap k^*) \to J_k^{\mathfrak{M}}/P_k^{\mathfrak{M}}$$

If $\alpha \in C_k^{\mathfrak{M}}$, then $(\alpha) = 1$, so $C_k^{\mathfrak{M}} \subseteq ker$. Conversely, if $[\alpha] \in ker$, $\alpha \in \mathbb{I}_k^{\mathfrak{M}}$, there is $(x) \in P_k^{\mathfrak{M}}$, $x \in \mathbb{I}_k^{\mathfrak{M}} \cap k^*$, such that $(\alpha) = (x)$. Consider $\beta = \alpha x^{-1}$, then if v is finite $\beta_v = 1 \bmod \wp_v^{n_v}$ by definition, and if v is real $\beta_v > 0$ since α_v and x_v are. Hence $\beta \in \mathbb{I}_k^{\mathfrak{M}}$, so $[\beta] \in C_k^{\mathfrak{M}}$, and since $[\beta] = [\alpha]$, we conclude. □

Let us suppose that k is a number field, $\mathcal{O}_k$ its ring of integers, $\mathcal{O}_k^\times \cong \mathbb{I}_k^{S_\infty} \cap k^\times$ the group of units and $(\mathcal{O}_k^\times)_{>0} = \mathbb{I}_k^1 \cap k^\times$ the group of totally positive units.

Proposition A.2.6. *There is an exact sequence of multiplicative abelian groups*

$$1 \longrightarrow \mathcal{O}_k^\times/(\mathcal{O}_k^\times)_{>0} \longrightarrow \textstyle\prod_{S_r} \mathbb{R}^\times/\mathbb{R}_{>0} \longrightarrow Cl_k^1 Cl_k \longrightarrow 1$$

Proof. Remark that $Cl_k^1 = C_k/C_k^1 = \mathbb{I}_k/\mathbb{I}_k^1 k^\times$ by proposition A.2.5 and $Cl_k = \mathbb{I}_k/\mathbb{I}_k^{S_\infty} k^\times$ by corollary A.1.18.
Proposition A.2.4 gives an exact sequence

$$1 \to \mathbb{I}_k^{S_\infty} k^\times/\mathbb{I}_k^1 k^\times \to C_k/C_k^1 \to Cl_k \to 1$$

and on the other hand we have an exact sequence

$$1 \to (\mathbb{I}_k^{S_\infty} \cap k^\times)/(\mathbb{I}_k^1 \cap k^\times) = \mathcal{O}_k^\times/(\mathcal{O}_k^\times)_{>0} \to \mathbb{I}_k^{S_\infty}/\mathbb{I}_k^1 = \prod_{S_r} \mathbb{R}^\times/\mathbb{R}_{>0} \to (\mathbb{I}_k^{S_\infty} k^\times)/(\mathbb{I}_k^1 k^\times) \to 1$$

So combining the two we have the result □

A.3 Extensions of the base field

Let now L/k be a finite separable extension, we have an embedding $\mathbb{I}_k \to \mathbb{I}_L$ given by

$$\alpha_v \mapsto \prod_{w|v}(\alpha_v)$$

Therefore, an element $\beta \in \mathbb{I}_L$ is in $\mathbb{I}_k$ if and only if $\beta_W \in k_v$ for all $w|v$ and if w_1 and w_2 divide v, then $\beta_{w_1} = \beta_{w_2}$.
In particular, we have that every element of $k^\times$ is in $L^\times$, therefore we have an induced map on the class group:

$$C_k \to C_L$$

which is injective since, f we fix M a normal closure of L with Galois group G:

$$\mathbb{I}_k \cap L^\times = \mathbb{I}_k \cap M^\times = (\mathbb{I}_k \cap M^\times)^G = \mathbb{I}_k \cap k^\times = k^\times$$

Every isomorphism $\sigma : L \to \sigma L$ induces an isomorphism $\mathbb{I}_L \to \mathbb{I}_{\sigma L}$ trivially since $\hat{\sigma} : L_w \to L_w$ is an isomorphism.
If now L/k is Galois with Galois group G, then every $\sigma \in G$ induces an automorphism of $\mathbb{I}_L$, hence $\mathbb{I}_L$ is a G-module. It is not difficult to show that we have the Galois descent for $\mathbb{I}_k$

$$\mathbb{I}_L^G = \{\alpha \in \mathbb{I}_L : \sigma\alpha = \alpha \text{ for all } \sigma \in G\} = \mathbb{I}_k$$

Moreover, we have the Galois descent for C_k:

Proposition A.3.1. *If L/k is Galois with Galois group G, then $C_L^G = C_k$*

Proof. We have an exact sequence

$$1 \to L^\times \to \mathbb{I}_L^\times \to C_L \to 1$$

And since $H^1(G, L^\times) = 0$ for Hilbert 90, the sequence

$$1 \to k^\times \to \mathbb{I}_k \to C_L^G \to 1$$

is exact, so $C_L^G = C_K$. □

Consider now v a place of k and $w|v$ a place of L. Then every α_w acts by multiplication on L_w, so we have a norm map

$$N_{L_w/k_v} : L_w \to k_v \quad N_{L_w/k_v}(\alpha_v) = det(\alpha_w(\cdot))$$

And since if $\alpha_w \in L_w^\times$, then $N_{L_w/k_v}(\alpha_v) \in k_w^\times$ and if $\alpha_w \in \mathcal{O}_w^\times$, then $N_{L_w/k_v}(\alpha_w) = 1$, the norm map extends to the idÃĺle:

$$N_{L/k}(\alpha) = \prod_{v \in S_k w|v} N_{L_w/k_v}(\alpha_w) : \mathbb{I}_L \to \mathbb{I}_k$$

The idelic norm has the same properties as the usual norm:

Proposition A.3.2. *i If $k \subseteq L \subseteq F$, then $N_{F/k} = N_{L/k}N_{F/L}$*

ii If L/k is embedded in a Galois extension F/k and if $G = Gal(F/k)$ and $H = Gal(F/L)$, then

$$N_{L/k}(\alpha) = \prod_{\sigma \in G/H} \sigma\alpha$$

iii if $\alpha \in \mathbb{I}_k$, then $N_{L/k}(\alpha) = \alpha^{[L:k]}$

iv If $x \in L^*$, then $N_{L/K}(x)$ is the usual norm

Proof. The proof of $i - iii$ is analogue to the case of the norm of a field extension and iv is immediate.

Remark A.3.3. Since by iv $N_{L/K}(L^\times) \subseteq k^\times$, we have an induced norm

$$N_{C_L/C_k} : C_L \to C_k$$

Lemma A.3.4. *For any modulus $\mathfrak{M} = \prod_{v \in S} \wp_v^{n_v}$ let*

$$k^{\times,\mathfrak{M}} := \{x \in k^\times : x = 1 \ mod \ \mathfrak{M}\}$$

If n_v is big enough we have

$$[K^\times : (N_{L/K}(L^\times))K^{\times\mathfrak{M}}] = \prod_{v \in S} \#G_v$$

Proof. For each v fix any $w|v$, since the extension is Galois, they all induce the same norm map. For local class field theory $N_{L_w/k_v}L_w \subseteq k_v$ is an open subgroup of finite index, hence it contains $1 + \wp_v^{n_v}$ if v is archimedean for a suitable n_p, or $\mathbb{R}_{>0}$ if v is real and $n_v = 1$. Hence define $\mathfrak{M} = \prod_{v \in S} \wp_v^{n_v}$.
Then consider $\mathfrak{M} = \prod_v \wp_v^{n_v}$ for the n_v just found. The natural map

$$k^\times/(N_{L/k}(L^\times))k^{\times\mathfrak{M}} \to \prod_{v:n_v \neq 0} k_v^\times/N_{L_w/k_v}(L_w^\times)$$

is bijective for weak approximation theorem: if $\alpha_v \in k_v^\times$, there is $x \in k^\times$ such that $x = \alpha_v$ mod $\wp_v^{n_v}$, so $x \mapsto (\alpha_v)_v$.
If $x \mapsto 0$, i.e. $x = (N_{L_w/k_v}L_w(\beta_v))_v$, then again by weak approximation there is $y \in L$ such that $y = \beta_v$ mod $\mathfrak{P}_w^{n_v}$, hence $x/N_{L/K}(y) \in K^{\times,\mathfrak{M}}$, so $x = 0$.
We conclude by local class field theory which says that

$$k_v^\times/N_{L_w/k_v}(L_w^\times) = [L_w : k_v] = \#G_v$$

□

A.4 Cohomology of the idÃĺle class group

In this section, we will prove that if k is a local field, then $C_{\overline{k}} = \varinjlim_{L/k} C_L$ is a class formation for the absolute Galois group G_k. We already defined Tate cohomology in Section 1.1.1

Proposition A.4.1. *Let L/k a Galois extension of degree n. Then*

- $\widehat{H}^1(G, C_L) = 0$
- $\#\widehat{H}^{2n}(G, C_L)$ *divides* n

Proof. See [CF67, VII.9]

So if L/k is a tower of extension, we have a commutative diagram given by the inflation-restriction exact sequence and passing to the limit:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^2(G(L/k), L^\times) \arrow[r] \arrow[d] & H^2(G(L/k), \mathbb{I}^{)}_L \arrow[r] \arrow[d] & H^2(G(L/k), C_L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(G_k, \overline{k}^\times) \arrow[r] \arrow[d] & H^2(G_k, \mathbb{I}_{\overline{k}}) \arrow[r] \arrow[d] & H^2(G_k, C_{\overline{k}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(G_L, \overline{k}^\times) \arrow[r] & H^2(G_L, \mathbb{I}_{\overline{k}}) \arrow[r] & H^2(G_L, C_{\overline{k}}) \arrow[r] & 0
\end{tikzcd}$$

and one can see ([CF67, VII.10]) that we have a complex

$$0 \to H^2(G(L/k), L^\times) \to H^2(G(L/k), \mathbb{I}_L) \xrightarrow{inv} \mathbb{Q}/\mathbb{Z}$$

where $inv = \sum_v inv_v$ where inv_v is the map defined in Section 1.1.3, and since $inv_v(infl(a)) = inv_v(a)$ the diagram commutes:

$$\begin{tikzcd}
H^2(G(L/k), \mathbb{I}_L) \arrow[r] \arrow[d, "infl"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "id"] \\
H^2(G_k, \mathbb{I}_L) \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Hence we have two complexes

$$0 \to H^2(G_k, \overline{k}^\times) \to H^2(G_k, \mathbb{I}_{\overline{k}}) \xrightarrow{inv} \mathbb{Q}/\mathbb{Z}$$

$$0 \to H^2(G_L, \overline{k}^\times) \to H^2(G_L, \mathbb{I}_{\overline{k}}) \xrightarrow{inv} \mathbb{Q}/\mathbb{Z}$$

and since $inv_w(res(\alpha)) = n_{w/v} inv_v(\alpha)$ and $\prod n_{w|v} = [L : k] = n$, we have a commutative diagram

$$\begin{tikzcd} H^2(G_k, \mathbb{I}_{\overline{k}}) \arrow[r] \arrow[d, "res"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "n"] \\ H^2(G_L, \mathbb{I}_{\overline{k}}) \arrow[r] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

and one can show ([CF67, VII:11.2]) that $H^2(G_k, C_{\overline{k}}) \cong \mathbb{Q}/\mathbb{Z}$ and it is induced by the arrows just defined, so the class formation axiom

$$\begin{tikzcd} H^2(G_k, \mathbb{I}_{\overline{k}}) \arrow[r, "\sim"] \arrow[d, "res"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "n"] \\ H^2(G_L, \mathbb{I}_{\overline{k}}) \arrow[r, "\sim"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

is satisfied.

Appendix B

Étale cohomology

B.1 Sheafification

If F is a presheaf on $(\mathcal{C}.\tau)$ a LEX site, consider a covering $\{U_i \to U\}$, we have the functor $\check{H}^0(\{U_i \to U\}, F)$ given by the kernel

$$\check{H}^0(\{U_i \to U\}, F) \rightarrowtail \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \times_U U_j)$$

It's functorial and left exact for the properties of the kernel. We can take the cofiltered category $Cov(U)/\sim$ given by the refinement condition and we have

$$F^+(U) := \check{H}^0(U, F) := \varinjlim_{Cov(U)/\sim} \check{H}^0(\{U_i \to U\}, F)$$

F^+ is a presheaf, in fact if $U \to V$, $\{U_i \to U\}$ a covering, then $\{U_i \times_U V \to V\}$ is a covering, we have the unique kernel morphism

$$\check{H}^0(U_i \times_U V \to V, F) \to \check{H}^0(U_i \to U, F)$$

induced by $F(U_i \times_U V) \to F(U_i)$ so for the universal properties of the filtered colimits we have a natural map

$$\check{H}^0(V, F) \to \check{H}^0(U, F)$$

We have now that if G is a sheaf, then by definition $G^+(U) = G(U)$, so if $F \xrightarrow{\alpha} G$ is a morphism of presheaves with G a sheaf, then for the properties of the kernel $\exists!$ map $F^+ \xrightarrow{\alpha^+} G$ such that the diagram commutes:

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[dr, "\epsilon"'] & & G \\ & F^+ \arrow[ur, "\alpha^+"'] & \end{tikzcd}$$

Recall that a presheaf is *separated* if the map

$$F(U) \rightarrowtail \prod F(U_i)$$

is a mono for all coverings. We have that

- F^+ is always separated
- If F is a separated, then $\forall$ $\{U_i \to U\}$ covering, $\forall$ $\{V_i \to U\}$ refinements, the map

$$\check{H}^0(U_i \to U, F) \to \check{H}^0(V_i \to U, F)$$

  is injective, hence $\check{H}^0(U_i \to U, F) \to F^+(U)$ is injective $\forall$ coverings
- If F is separated, then F^+ is a sheaf

Proof. [Sta, Tag 00WB]
So we have a functor

$$a : Psh(\mathcal{C}) \to Sh(\mathcal{C}, \tau) \qquad F \mapsto F^{++}$$

and for the property above

$$\mathrm{Hom}_{Psh}(F, iG) \cong \mathrm{Hom}_{Sh}(aF, G) \qquad \alpha \mapsto \alpha^{++}$$

So $a \dashv i$. So we have that

$$Sh(\mathcal{C}, \tau) \overset{a}{\underset{}{\leftrightarrows}} Psh(\mathcal{C})$$

is a reflective subcategory, so if D is a diagram in $Sh(\mathcal{C}, \tau)$ such that iD has a limit L in $Psh(\mathcal{C})$, then aL is a limit of D in $Sh(\mathcal{C}, \tau)$.
So since $(\_)^{++}$ is left exact as endofunctor of $Psh(\mathcal{C})$, a is left exact.

B.1.1 Yoneda

Consider $y : \mathcal{C} \to Sh(\mathcal{C}, \tau)$ the sheafification of the Yoneda embedding. Take F a sheaf, $X \in \mathcal{C}$. Then, we have

$$F(X) \cong \mathrm{Hom}_{Psh(\mathcal{C})}(h_X, F) \cong \mathrm{Hom}_{Sh(\mathcal{C})}(yX, F)$$

Lemma B.1.1. *If $U_i \to X$ is a covering, then the induced map*

$$\coprod yU_i \to yX$$

is an epimorphism.

Proof. Consider $U_i \to X$ By the sheaf property, we have

$$FX \hookrightarrow \prod FU_i$$

is a regular mono. Applying Yoneda lemma and the remark above, we get

$$\mathrm{Hom}_{Sh(\mathcal{C})}(y_X, F) \hookrightarrow \prod_i \mathrm{Hom}_{Sh(\mathcal{C})}(yU_i, F) = \mathrm{Hom}_{Sh(\mathcal{C})}(\coprod_i yU_i, F)$$

is a mono, so

$$\mathrm{Hom}_{Sh(\mathcal{C})}(y_X, \_) \to \mathrm{Hom}_{Sh(\mathcal{C})}(\coprod_i yU_i, \_)$$

is a monomorphism of representable covariant functors in $Set^{Sh(\mathscr{C})}$, and since the Yoneda emabedding is fully faithful and countervariant, it reflects monos to epis (and viceversa), so

$$\coprod yU_i \twoheadrightarrow yX$$

is an epi $\square$

Proposition B.1.2. *Consider $F \xrightarrow{f} G$ a morphism of sheaves. If $\forall\ X \in \mathscr{C},\ \forall\ b \in G(X)$ we have that $\exists\ \{U_i \to X\}$ a covering and $a_i \in F(U_i)$ such that*

$$f(a_i) = \beta_{|U_i}$$

Then f is an epi of sheaves.

Proof. Using the isomorphism above, we get $a_i : yU_i \to F$ and $b : yX \to G$ Take $s, t : G \overset{\to}{\to} H$ such that $sf = tf$. We have

$$\begin{tikzcd}
\coprod yU_i \arrow[r, two heads, "\phi"] \arrow[d, "(a_i)"] & yX \arrow[d, "b"] \arrow[dr, shift left, "r(b)"] \arrow[dr, shift right, "s(b)"'] & \\
F \arrow[r, "f"] & G \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & H
\end{tikzcd}$$

The central square commutes by hypothesis and the right triangle commutes by construction, so we have

$$s(b)\phi = sf(a_i) = rf(a_i) = r(b)\phi$$

And since ϕ is an epi, we have that $\forall\ X,\ \forall\ b \in G(X)\ r_X(b) = s_X(b) \Rightarrow r = s$ $\square$

B.2 Direct images

Let $f : \mathscr{C} \to \mathscr{D}$ a functor between categories. Consider:

$$\begin{aligned} f_p : & Psh(\mathscr{D}) \to Psh(\mathscr{C}) \\ & F \mapsto F(f()) \end{aligned}$$

Proposition B.2.1. *f_p has a left adjoint f^p*

Proof. Fix $X' \in \mathscr{D}$. Consider the category $I_{X'}$ given by

$$\{(X, \phi), X \in \mathscr{C}, \phi : X' \to f(X)\}$$

and arrows given by $X_1 \xrightarrow{g} X_2$ such that the following diagram commutes:

$$\begin{tikzcd}
f(X_1) \arrow[rr, "f(g)"] & & f(X_2) \\
& X' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Consider $F \in Psh(\mathcal{C})$

$$f^p(F)(X') := \operatorname*{colim}_{T \in I_{X'}} F(T)$$

Such a colimit exists since Set and Ab are cocomplete. We have that $(X, id) \in I_{f(X)}$, so we have

$$f_p f^p(F)(X) = \operatorname*{colim}_{T \in I_{f(X)}} \xleftarrow{\exists!} F(X)$$

And we have that by definition if $X \in I_{X'}$ we have an arrow $F(f(X)) \to F(X')$ (F is countervariant) so we have

$$f^p f_p(F)(X') = \operatorname*{colim}_{T \in I_{X'}} F(f(x)) \xrightarrow{\exists!} F(X')$$

and the triangular identities are given by the universal properties. □

Proposition B.2.2. *In the notation above, if $\mathcal{C}$ is LEX and f is LEX, then f^p is exact*

Proof. By definition, if $I_{X'}$ is cofiltered (it becomes filtered applying F), we have that $\varinjlim_{I_{X'}}$ is exact, so it's enough to prove that it's cofiltered. We have

1. $I_{X'} \neq \emptyset$ since if T is the terminal object of $\mathcal{C}$, $f(T)$ is the terminal object of $\mathcal{D}$ and we have $X \to f(T)$ the only map to the terminal.
2. $(X_1, \phi_1), (X_2, \phi_2)$, consider $X_1 \times X_2$, we have $f(X_1 \times X_2) = f(X_1) \times f(X_2)$ and

$$\begin{tikzcd} & X \arrow[d, dotted, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & f(X_1) \times f(X_2) \arrow[dl] \arrow[dr] & \\ f(X_1) & & f(X_2) \end{tikzcd}$$

The diagram commutes so it's cofiltered.

□

Take now f a continuous LEX functor, i.e. $\forall \{U_i \to U\}$ covering, then $f(U_i) \to f(U)$ is a covering. This trivially means that if $F \in Sh(\mathcal{D})$, then $f_*F := F(f) \in Sh(\mathcal{C})$ so we have a functor:

$$f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$$

such that $i'f_* = f_p i$
We have the following diagram:

$$\begin{tikzcd} Psh(\mathcal{C}) \arrow[r, shift left, "f^p"] \arrow[d, shift left, "a"] & Psh(\mathcal{D}) \arrow[l, shift left, "f_p"] \arrow[d, shift left, "a'"] \\ Sh(\mathcal{C}, \tau) \arrow[u, shift left, "i"] \arrow[r, shift left, "f^*"] & Sh(\mathcal{D}, \tau) \arrow[u, shift left, "i'"] \arrow[l, shift left, "f_*"] \end{tikzcd}$$

taking $f^* := a' f^p i$, it's left exact since it's composition of left exact functors:

Proposition B.2.3. *$f^* \dashv f_*$ and so f^* is exact.*

Proof. The adjunction follows trivially by $f^p \dashv f_p$ and the fully faithfulness of the inclusion:

$$Hom_{Sh(\mathcal{D})}(f^*F, G) \cong \mathrm{Hom}_{Psh(\mathcal{D})}(f^p iF, i'G) \cong$$
$$Hom_{Psh(\mathcal{C})}(iF, f_p i'G) \cong \mathrm{Hom}_{Psh(\mathcal{C})}(iF, if_*G) = Hom_{Sh(\mathcal{C})}(F, f_*G)$$

□

Lemma B.2.4. *Comparison Lemma*
If $f : \mathcal{C}' \hookrightarrow \mathcal{C}$ is a LEX fully faithful inclusion of LEX sites such that:

1. *If $\{U_i \to U\} \in Cov_{\mathcal{C}'}(U)$, $U,\ U_i \in \mathcal{C}'$, then $\{U_i \to U\} \in Cov_{\mathcal{C}}(U)$*
2. *$\forall\ U' \in \mathcal{C}\ \exists\ \{U_i \to U'\} \in Cov_{\mathcal{C}'}(U')$ with $U_i \in \mathcal{C}$*

Then f^* and f_* are quasi inverse and induce an equivalence of categories

Proof. We need to show that the unit and the counit are natural isomorphisms:

1. Take $F \in Sh(\mathcal{C}')$, $U \in \mathcal{C}'$, we have $f_*f^*F(U) = (f^pF)^{\#}(f(U))$, so we need to prove that

$$F(U) \xrightarrow{(i)} f^pF(f(U)) \xrightarrow{(ii)} f_*f^*F(U)$$

are isomorphisms:

(i) $f^pF(f(U)) = \varinjlim_{I_{f(U)}} F(X)$
We have $(U, id) \in I_{f(U)}$, and since f is fully faithful, $\forall\ \phi : f(U) \to f(X)$ we have that $\exists!\ \psi : U \to X$ such that $\phi = f(\phi)$, (U, id) is the initial object, so the unique map

$$F(U) \to \varinjlim_{I_{f(U)}} F(X)$$

is an iso.

(ii) By the property *2.* we have that if $U \in \mathcal{C}'$, then $Cov_{C'}(U) \hookrightarrow Cov_C(U)$ is cofinal, so

$$f^*F(U) = \varinjlim_{Cov_C(U)} \check{H}^0(U_i \to U, f^pF) = \varinjlim_{Cov_{C'}(U)} \check{H}^0(U_i' \to U, f^pF)$$

since now $f^pF(U) = F(U)$ for the previous point

$$\varinjlim_{Cov_{C'}(U)} \check{H}^0(U_i' \to U, f^pF) = \varinjlim_{Cov_{C'}(U)} Ker(\prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j))$$

which is trivially $F(U)$ since F is a sheaf.

2. Take $F \in Sh(\mathcal{C})$, $U \in \mathcal{C}$. Using the triangular identity, we have that if $U \in \mathcal{C}'$, then $\epsilon_{F(U)}$ is an isomorphism, in fact, since $U = f(U)$ by the fully faithfulness, the triangular identity is

$$\begin{tikzcd} f_*f^*F(U) \arrow[r, "\epsilon_{F(U)}"] & F(U) \\ F(U) \arrow[u, "\eta_U"] \arrow[ur, equal] & \end{tikzcd}$$

and η is a natural iso.
Now if $U \in \mathcal{C}$, take $\{U_i \to U'\}$ as in 2., we have

$$\begin{tikzcd} 0 \arrow[r] & FU \arrow[r] & \prod F(U_i) \arrow[r] & \prod F(U_i \times_U U_j) \\ 0 \arrow[r] & f_*f^*FU \arrow[r] \arrow[u] & \prod f_*f^*F(U_i) \arrow[r] \arrow[u, "\sim"] & \prod f_*f^*F(U_i \times_U U_j) \arrow[u, "\sim"] \end{tikzcd}$$

So since the right-hand square commutes, the arrows on the kernels is an isomorphism.

□

B.3 Cohomology on a site

Proposition B.3.1. *If $(\mathcal{C}, \tau)$ is a site, then $Psh_{Ab}(\mathcal{C})$ and $Sh_{Ab}(\mathcal{C})$ have enough injectives.*

Proof. see [Sta, Tag 01DK, Tag 01DL]

B.3.1 Čech cohomology

Let $(\mathcal{C}, \tau)$ be a site, $\{U_i \to U\}_I$ a covering, F an abelian presheaf. Consider the Čech complex:

$$C^k(\{U_i \to U\}_I, F) := \prod_{(i_0 \dots i_k) \in I^{k+1}} F(U_{i_0} \times_U \dots \times_U U_{i_k})$$

with cobords maps given by

$$d^k(a)_{i_0 \dots i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j (a_{i_0 \dots \hat{j} \dots i_{k+1}})_{|U_{i_0} \times_U \dots \times_U U_{i_{k+1}}}$$

One can check that this is in fact a complex.
We can take its cohomology

$$\check{H}^q(U_i \to U, F) = H^q(C^\bullet(U_i \to U))$$

By definition, if F is a sheaf, then

$$\check{H}^0(U_i \to U, F) = F(U)$$

Proposition B.3.2. *$\check{H}^q(U_i \to U, F) \cong R^q(H^0(U_i \to U, \_))(F)$, i.e. $\check{H}^q(U_i \to U, \_)$ is a universal cohomological δ-functor.*

Proof. One can see that the Čech complex preserves exact sequences of *presheaves*, so $\check{H}^q(U_i \to U, \_)$ is a cohomological δ-functor on $Psh(\mathcal{C})$. We need to show that injectives are acyclics: Take $X \in \mathcal{C}$ consider the free functor:

$$\mathbb{Z}_X(Y) = \mathbb{Z}^{Hom_{\mathcal{C}}(X,Y)}$$

We have that the free functor is left adjoint to the forgetful, so $\forall\ X,\ Y$

$$Hom_{Ab}(\mathbb{Z}^{Hom_{\mathcal{C}}(X,Y)}, F(Y)) \cong \mathrm{Hom}_{Set}(Hom_{\mathcal{C}}(X,Y), F(Y))$$

natural in Y, so

$$Hom_{Psh_{Ab}(\mathcal{C})}(\mathbb{Z}_X, F) \cong \mathrm{Hom}_{Psh(\mathcal{C})}(h_X, F) \cong F(X)$$

So we have

$$C^q(U_i \to U, F) \cong \prod_{(i_0 \dots i_q)} \mathrm{Hom}_{Psh_{Ab}(\mathcal{C})}(\mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_q}}, F) \cong \mathrm{Hom}_{Psh_{Ab}(\mathcal{C})}(\bigoplus_{(i_0 \dots i_q)} \mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_q}}, F)$$

So if I is an injective presheaf, $Hom_{Psh_{Ab}(\mathcal{C})}(\_, I)$ is an exact functor,

$$\dots \bigoplus_{(i_0 \dots i_{q-1})} \mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_{q-1}}} \to \bigoplus_{(i_0 \dots i_q)} \mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_q}} \dots$$

is an exact complex, so

$$\dots Hom_{Psh_{Ab}(\mathcal{C})}(\bigoplus_{(i_0 \dots i_{q-1})} \mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_{q-1}}}, I) \to \mathrm{Hom}_{Psh_{Ab}(\mathcal{C})}(\bigoplus_{(i_0 \dots i_q)} \mathbb{Z}_{U_{i_0} \times_U \dots \times_U U_{i_q}}, I) \dots$$

is an exact complex, so the Čech complex is exact. $\square$

One can consider again the refinement equivalence relation on $Cov(U)$, and get again that if $\{U_i \to U\}$ and $\{V_j \to U\}$ mutually refines, then

$$\check{H}^q(U_i \to U, F) = \check{H}^q(V_j \to U, F)$$

So one can define

$$\check{H}^q(U, F) := \varinjlim_{Cov(U)/\sim} (\check{H}^q(U_i \to U, F))$$

And since the colimit is filtered, it's exact, so we have a long exact sequence. Moreover, if $\check{H}^q(U_i \to U, I)) = 0\ \forall\ I$ injectives and $\forall\ \{U_i \to U\}$ coverings, we have $\check{H}^q(U, I) = 0$, so

$$\check{H}^q(U, F) \cong R^q(\check{H}^0(U, \_))(F)$$

B.3.2 Cohomology of Abelian sheaves

Let $(\mathcal{C}, \tau)$ be a Lex site, $X \in \mathcal{C}$. We have a left-exact functor

$$\Gamma_X : Sh_{Ab}(\mathcal{C}, \tau) \to Ab$$

So we can derive it and get:

$$H^q(X, F) := R^q(\Gamma_X)(F)$$

Consider now

$$i : Sh(\mathcal{C}) \hookrightarrow Psh(\mathcal{C})$$

the inclusion functor, which is right adjoint, so left exact. Consider

$$\mathcal{H}^q(F) = R^q(i)(F) \in Psh(\mathcal{C})$$

In fact, since any arrow $U \to V$ gives a natural map $\Gamma_V \to \Gamma_U$, we have $H^0(\_, F) = F$. If

$$F' \to F \to F''$$

is exact, then

$$H^q(X, F'') \to H^{q+1}(X, F')$$

is natural in X, so we have a long exact sequence of presheaves, and moreover if I is injective, then $H^q(X, I) = 0\ \forall\ X$, so $H^q(\_, I) = 0$, so

$$\mathcal{H}^q(F) = H^q(\_, F)$$

Proposition B.3.3. $(\mathcal{H}^q(F))^+ = 0\ \forall q \geq 1$

Proof. In fact, i has an exact left-adjoint a, so it preserves injectives[1], so we can use Grothendieck's Theorem $(ai = id_{Sh})$:

$$R^p(a)R^q(i) \Rightarrow R^{p+q}(id)$$

But a is exact $\Rightarrow$ the SS degenerates at degree 2, so since id is exact

$$H^q(F)^\# = R^q(id) = 0$$

So now we have the counit maps

$$H^q(F) \to H^q(F)^+ \hookrightarrow H^q(F)^\#$$

the second is mono since $H^q(F)^+$ is separated, so $H^q(F)^+ = 0$ □

Theorem B.3.4. *Let F be a sheaf. We have two spectral sequences*

$$\check{H}^p(U_i \to U, H^q(F)) \Rightarrow H^{p+q}(U, F)$$

$$\check{H}^p(U, H^q(F)) \Rightarrow H^{p+q}(U, F)$$

[1] $\mathrm{Hom}(\_, iI) = \mathrm{Hom}(a(\_), I)$, a is exact, $\mathrm{Hom}(\_, I)$ is exact since I is injective

Proof. Since if F is a sheaf, $\check{H}^0(U_i \to U, F) = \check{H}^0(U, F) = F(U)$, so we have

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, hook, "i"] \arrow[rr, bend right, "\Gamma_U"'] & Psh(\mathcal{C}) \arrow[r, "\check{H}^0(U_i \to U\_)"] & Ab \end{tikzcd}$$

And again since i preserves injectives, we have a SS

$$R^p(\check{H}^0(U_i \to U, \_))R^q(i)(F) \Rightarrow R^{p+q}(\Gamma_U)(F)$$

□

Corollary B.3.5. *We have:*

1. $\check{H}^0(U, F) \cong H^0(U, F)$
2. $\check{H}^1(U, F) \cong H^1(U, F)$
3. $\check{H}^2(U, F) \rightarrowtail H^2(U, F)$

Proof. *1.* is trivial. *2.* and *3.* follow directly from the exact sequence of low degree terms:

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^1(U, F) \arrow[r] & H^1(U, F) \arrow[r] & \check{H}^0(U, H^1F) = 0 \arrow[dll, rounded corners] \\ & \check{H}^2(U, F) \arrow[r] & H^2(U, F) \arrow[r] & \dots \end{tikzcd}$$

□

B.3.3 Flasque Sheaves

Definition B.3.6. $F \in Sh(\mathcal{C})$ is *Flasque* (or *Flabby*) if for all U, for all $\{U_i \to U\}$ we have $\check{H}^q(U_i \to U, F) = 0, \quad q > 0$

It's clear that if F is an injective sheaf, then since i preserves injectives and $\check{H}^q(U_i \to U, \_)$ is a universal δ-functor, then F is Flabby

Proposition B.3.7. *Consider* $0 \to F' \to F \to F'' \to 0$ *exact sequence of sheaves:*

i. *If F' is flasque, then it's an exact sequence of presheaves*

ii. *If F' and F are flasque, then F'' is flasque.*

iii. *If F and G are flasque, then $F \oplus G$ is flasque*

Proof. i. Consider the long exact sequence

$$\begin{tikzcd} \check{H}^0(U_i \to U, F') \arrow[r] & \check{H}^0(U_i \to U, F) \arrow[r] & \check{H}^0(U_i \to U, F'') \arrow[dll, rounded corners] \\ \check{H}^1(U_i \to U, F') = 0 & & \end{tikzcd}$$

And since they are all sheaves, we have $0 \to F'(U) \to F(U) \to F''(U) \to 0$ exact $\forall\ U$.

ii. Consider the long exact sequence

$$\check{H}^q(U_i \to U, F) \longrightarrow \check{H}^q(U_i \to U, F'') \longrightarrow \check{H}^{q+1}(U_i \to U, F')$$

If $q \geq 1$, then

$$\check{H}^q(U_i \to U, F) = \check{H}^{q+1}(U_i \to U, F') = 0 \Rightarrow H^q(U_i \to U, F'') = 0$$

iii. Since

$$\prod_{i_0 \dots i_q} (F \oplus G)(U_{i_0} \times_U \dots \times_U U_{i_q}) \cong \prod_{i_0 \dots i_q} F(U_{i_0} \times_U \dots \times_U U_{i_q}) \oplus \prod_{i_0 \dots i_q} G(U_{i_0} \times_U \dots \times_U U_{i_q})$$

we have an isomorphism of complexes

$$C^\bullet(U_i \to U, F \oplus G) \cong C^\bullet(U_i \to U, F) \oplus C^\bullet(U_i \to U, G)$$

And so

$$\check{H}^q(U_i \to U, F \oplus G) \cong \check{H}^q(U_i \to U, F) \oplus \check{H}^q(U_i \to U, G) = 0 \text{ for } q \geq 1$$

□

Corollary B.3.8. *If F is a sheaf, then TFAE:*

i. F is flasque

ii. F is $\mathcal{H}^q$-acyclic (so $H^q(U, F) = 0$ for all U)

Proof. *i.* $\Rightarrow$ *ii*. Consider an injective resolution

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] & I_1 \arrow[rr] \arrow[dr] & & I_2 \dots \\ & & & I_1/F \arrow[ur] & \end{tikzcd}$$

Since F and I_1 are flasque, then I_1/F is flasque, and by induction if I_1/I_{j-1} and I_{j+1} are flasque then I_{j+1}/I_j is flasque, so

$$\begin{tikzcd} 0 \arrow[r] & iF \arrow[r] & iI_1 \arrow[rr] \arrow[dr] & & iI_2 \dots \\ & & & i(I_1/F) \arrow[ur] & \end{tikzcd}$$

is exact in $Psh(\mathcal{C})$, so $\mathcal{H}^q(F) = R^q(i)(F) = 0$

ii. $\Rightarrow$ *i.* Consider the spectral sequence

$$\check{H}^q(U_i \to U, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F)$$

It degenerates in degree 2 by hypothesis, so

$$\check{H}^q(U_i \to U, F) \cong H^p(U, F) = 0$$

□

B.3.4 Higher Direct Image

If $(\mathcal{C}, \tau_C) \xrightarrow{f} (\mathcal{D}, \tau_D)$ is a morphism of LEX site, we have

$$Sh(\mathcal{D}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} Sh(\mathcal{C})$$

with $f^* \dashv f_*$ and f^* exact, so f_* preserves injectives.
If now $T' \xrightarrow{f} T \xrightarrow{g} T''$ are morphisms of LEX sites, by definition

$$(gf)_*(F) = F(gf) = f_*F(g) = (f_*g_*)(F)$$

So since g_* preserves injectives we have a spectral sequence

$$R^p f_* R^q g_*(F) \Rightarrow R^{p+q}(gf)_*(F)$$

Remark B.3.9. $f_* = af_pi'$,since i' preserves injectives and a and f_p are exact [2], we have a spectral sequence degenerating at degree 2

$$R^p(af_p)R^q(i')(F) \Rightarrow R^{p+q}f_*F$$

So $R^q f_*F \cong (f_p\mathcal{H}^q(F))^\#$

Corollary B.3.10. *If F is flasque, then $R^q f_*F = 0 \ \forall \ q \geq 1$*

If e_T the terminal object of a LEX site T, we set

$$H^p(T, F) := H^p(e_T, F)$$

Remark B.3.11. If $f : T \to T'$ morphism of LEX sites, e_T the terminal object of T, we have $f(e_T) = e'_T$, so $(\Gamma_{e_T} f_*)(F) = F(f(e_T)) = \Gamma_{e_{T'}}$, so we have Leray's spectral sequence

$$H^p(T, R^q f_*F) \Rightarrow H^{p+q}(T', F)$$

This gives a canonical map

$$H^p(T, f_*F') \to H^p(T', F')$$

And if we consider $F \to f_*f^*F$ the unit map, we have a canonical map

$$H^p(T, F) \to H^p(T, f_*f^*F) \to H^p(T', f^*F)$$

B.4 Étale cohomology

B.4.1 The small Ãľtale site

Definition B.4.1. Let k be a field, a finite k-algebra A is **Ãľtale** if

$$A \cong K_1 \times ... \times K_n$$

with K_j/k finite separable.

[2] $0 \to F' \to F \to F'' \to 0$ exact of $Psh(\mathcal{C})$ iff $0 \to F'(X) \to F(X) \to F''(X) \to 0$ is exact $\forall$ X, in particular if $X = fY$

Consider X a Noetherian scheme.
Consider a morphism of finite type $Y \to X$, $y \in Y$, $x = f(y)$

Definition B.4.2. f is **unramified at** $x \in X$ if the schematic fiber $Y_x = Y \times_X Spec(k(x))$ is affine, namely $Y_x = Spec(B)$, and B is an Ãl'tale $k(x)$-algebra.

Definition B.4.3. A morphism of finite type $Y \to X$ is **unramified at** $y \in Y$ if $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ satisfies

$$\mathfrak{M}_x \mathcal{O}_{Y,y} = \mathfrak{M}_y$$

(i.e. it's of *relative codimension* 0) and $k(x) \subseteq k(y)$ is separable.

Remark B.4.4. If f is unramified at y, we have

$$Y_x = \coprod_{y_j \in f^{-1}(x)} Spec(k(y_j)) = Spec(\prod_{y_j \in f^{-1}(x)} k(y_j))$$

Since locally

$$(Y_x)_{y_j} \cong Spec(\mathcal{O}_{Y,y_j} \otimes_{\mathcal{O}_{X,x}} K(x)) \cong Spec(\mathcal{O}_{Y,y_j}/\mathfrak{M}_{y_j})$$

So if f is unramified at y, it's unramified at $f(y)$

Definition B.4.5. A morphism of finite type $Y \to X$ is **Ãl'tale at** $y \in Y$ if it's flat and unramified
f is **Ãl'tale** if it's Ãl'tale $\forall\ y \in Y$

We have trivially the following properties:

1. Open immersions are Ãl'tale, closed immersions are unramified
2. The composition of unramified (Ãl'tale) is unramified (Ãl'tale)
3. Base change of unramified (Ãl'tale) is unramified (Ãl'tale)

We have the following:

Lemma B.4.6. *Let $S \xrightarrow{f} S$ a morphism of finite type, then TFAE:*

i. f is unramified

ii. $\Delta_{X/S} : X \to X \times_S X$ is an open immersion

Proof. See [Sta], Lemma 28.33.13 □

So we have the following

Proposition B.4.7. *If $f : Y \to X$, $g : Z \to Y$ such that fg is Ãl'tale and f is Ãl'tale, then g is Ãl'tale*

Proof. Consider the base change

$$\begin{tikzcd} Y \arrow[r, "(g\ id)"] \arrow[d, "g"] & X \times_S Y \arrow[d, "(id\ g)"] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

Since Δ is open for the previous lemma, $Y \to X \times_S Y$ is Ãľtale. Consider then

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d, "g"] & Y \arrow[d, "fg"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

By definition, $X \times_S Y \to Y$ is the second projection, so it's Ãľtale since f is Ãľtale. Finally $Y \xrightarrow{(id,g)} Y \times_S X \xrightarrow{\pi_2} X$ is Ãľtale. $\square$

Consider $Et(X)$ the category of the étale X-schemes, we can define the **étale topology** τ_{et} on $Et(X)$ as

$$\{U_i \xrightarrow{f_i} U\} : U = \bigcup_i f_i(U_i)$$

By the previous proposition, this maps are all étale and the topology is subcanonical.

Definition B.4.8. We define the **small Ãľtale site** of X

$$X_{et} = \{Et(X), \tau_{et})\}$$

So we can define $\forall\ F \in Sh(X_e t)$ the **étale cogomology** of F as

$$H^p_{et}(X, F) = R^p(\Gamma_X)(F)$$

If $X' \in X_{et}$, we define

$$H^p_{et}(X', F) = R^p(\Gamma_{X'})(F)$$

Proposition B.4.9. *Consider $Y \xrightarrow{f} X$ a morpfism of schemes, we have a morphism of LEX sites*

$$X_{et} \xrightarrow{f^{-1}} Y_{et} \qquad (X' \to X) \mapsto (X' \times_X Y \to Y)$$

Proof.

- Since $X' \to X$ is étale and the fiber product of étale is étale, $f^{-1}(X')$ is étale.
- It is left exact by definition: it preserves final object ($X \times_X Y \cong Y$) and fiber product (universal properties and diagram chasing).
- Consider $\{U_i \xrightarrow{f} U\}$ an étale covering, we need $U \times_X Y = \cup_i (f_i \times_X id)(U_i \times_X Y)$, i.e. we need

$$\coprod_i (U_i \times_X Y) \xrightarrow{(f_i \times_X id)} U \times_X Y$$

to be an epimorphism.
Since $\forall\ u \in U\ \exists i : \exists\ u_i \in U_i : u = f_i(u_i)$, using Yoneda $\forall\ \Omega$ algebraically closed fields $\exists\ i$ such that the following diagram commutes:

$$\begin{tikzcd}
 & U_i \arrow[d, "f_i"] \\
Spec(\Omega) \arrow[ur, bend left, "u_i"] \arrow[r, "U"] & U
\end{tikzcd}$$

Consider $u' \in U \times_X Y$ such that $\pi_2(u') = u = f_i(u_i)$, we have

$$U_i \times_X Y \cong U_i \times_X Y \times_U U \cong (U \times_X Y) \times_U U_i$$

so for the universal property of the fiber product

$$\begin{tikzcd}
Spec(\Omega) \arrow[dr, "\exists!\ u_i'"] \arrow[drr, bend left, "u_i"] \arrow[ddr, bend right, "u'"'] & & \\
 & U_i \times_X Y \arrow[r] \arrow[d] & U_i \arrow[d] \\
 & U \times_X Y \arrow[r] & U
\end{tikzcd}$$

So f^{-1} is continuous.

$\square$

So if $Y \xrightarrow{f} X$ is a morphism of schemes, it induces a morphism of topoi

$$Sh(Y_{et}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} Sh(X_{et})$$

With $(f_*F)(X') = F(X' \times_X Y)$ and $f^*(F') = (f^p(F'))^{\#}$. We have that

$$(f^p(F'))(Y') = \varinjlim_{(X',\phi)\in I_{Y'}} F(X')$$

Where X'/X is étale and the diagram commutes

$$\begin{tikzcd}
 & Y' \arrow[dl, bend right, "\phi"'] \arrow[d] \\
X' \times_X Y \arrow[r] & Y
\end{tikzcd}$$

So by the universal property, we just get the pairs (X', ψ) where

$$\begin{tikzcd}
Y' \arrow[r, "\psi"] \arrow[d] & X' \arrow[d] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

If now f is étale, we have (Y', Id) is the initial object of $I_{Y'}$, hence

$$f^pF(Y') = F(Y')$$

Hence $f^pF = F_{|Y}$, and it's a sheaf, so $f^*F = F_{|Y}$ One can show ([Sta], Lemma 18.16.3) that since f is étale, we have that f^* has an exact left adjoint $f_!$, so f^* preserves injectives. So we have a composition

$$\begin{tikzcd} Sh(X_{et}) \arrow[r, "f^*"] \arrow[rr, bend right, "\Gamma_Y"'] & Sh(Y_{et}) \arrow[r, "\Gamma"] & \mathcal{A}b \end{tikzcd}$$

So we have a spectral sequence

$$R^p(\Gamma)R^q(f^*)F \Rightarrow R^{p+q}(\Gamma_Y)F$$

But f^* is exact, so it degenerates in degree 2 and we have

$$H^p(X_{et}, F_{|X}) \cong H^p(Y_{et}, X, F)$$

On the other hand,

$$(R^qf_*F)(X') = H^q(Y_{et}, Y \times_X X', F) \cong H^q((Y \times_X X')_{et}, F)$$

Again we have Leray SS

$$H^p(X, R^qf_*F) \Rightarrow H^{p+q}(Y, F)$$

gives the maps

$$H^p(X, f_*F) \to H^p(Y, F) \tag{B.1}$$

And again if $F = f^*G$, we have $G \to f_*f^*G$, then

$$H^p(X, G) \to H^p(X, f_*f^*G) \to H^p(Y, f^*G) \tag{B.2}$$

B.5 Galois cohomology

B.5.1 G-modules

Definition B.5.1. If G is a topological group, let $G - set$ be the category of continuous G-sets, i.e. G-sets where

$$G \times X \to X$$

Is continuous if we endow X with the discrete topology.

Proposition B.5.2. *If G is a profinite group, then we have an equivalence of categories*

$$Gset \longrightarrow Sh(Gset^f, \tau_c)$$

$$X \longmapsto Hom_G(\_, X)$$

Proof.

1. $\mathrm{Hom}_G(\_, X)$ is a sheaf for the canonical topology (i.e. the functor is well defined)

   - If X is finite, then by definition the canonical topology is the finest where representables are sheaves.
   - If $X \in Gset$, then consider the stabilizer
   $$G_x = \{g \in G : gx = x\}$$
   G_x is the fiber of $\{x\}$, open since $\{x\}$ is open. In particular, $\#O_x = [G : G_x]$ is finite, so
   $$X = \coprod_i X_i$$
   for X_i finite sets
   - $\mathrm{Hom}_G(\_, \coprod X_i) \cong \coprod \mathrm{Hom}_G(\_, X_i)$, and the coproduct of sheaves is a sheaf

2. Define a quasi-inverse:
   Let $H \leq G$ open normal subgroup, define $E(G/H)$ the continuous left G-set defined by the action $\sigma\bar{x} = \bar{\sigma}\bar{x}$
   H acts trivially on $E(G/H)$, so $G_g = Hg^{-1}$, open, so the action is continuous. We have a right action that gives a map of G-sets $(\_\sigma)$, so if F is a sheaf we have an action over $F(E(G/H)$ given by $\sigma x = F(\_\sigma)(x)$. Since if $H' \leq H$ is normal in G we have
   $$F(E(G/H)) \to F(E(G/H'))$$
   a functor
   $$Sh(Gset^f, \tau_C) \longrightarrow G - set$$
   $$F \longmapsto \mathrm{colim}_H\, F(E(G/H))$$

3. Consider the canonical isomorphism
   $$\psi_1 : \mathrm{Hom}_G(E(G/H), Z) \to Z^H$$
   Then we have
   $$\mathrm{colim}_H \mathrm{Hom}_G(E(G/H), Z) \cong \mathrm{colim}_H Z^H \cong Z$$

4. since $\{E(G/H_0) \xrightarrow{x} X^{H_0}\}_x$ is a covering for the canonical topology, we have
   $$F(X^{H_0}) \to \prod_x F(E(G/H_0)) \rightrightarrows \prod_{(x,x)} F(E(G/H_0) \times_{X^{H_0}} E(G/H_0))$$

is exact
$$\prod_x F(E(G/H_0)) = \mathrm{Hom}_{Set}(X^{H_0}, F(E(G/H_0)))$$
and $s \in Ker(\rightrightarrows)$ iff s is G/H_0-equivariant, so $F(X^{H_0}) \cong \mathrm{Hom}_{G/H_0}(X^{H_0}, F(E(G/H_0)))$
Take now H_0 small enough such that $X^{H_0} = X$ (it is finite), so

$$\begin{tikzcd}
F(X) = F(X^{H_0}) \arrow[r, "\sim"] & \mathrm{Hom}_{G/H_0}(X^{H_0}, F(E(G/H_0))) \arrow[d, "\sim"] \\
\mathrm{Hom}_G(X, F(E(G/H_0))) & \mathrm{Hom}_G(X, F(E(G/H_0)^{H_0})) \arrow[l, "\sim"']
\end{tikzcd}$$

□

Remark B.5.3. The same proof gives an equivalence
$$Gmod \cong Sh_{Ab}(Gset^f, \tau_C)$$
$$M \to \mathrm{Hom}_G(\_, M)$$
So if e is the terminal object in $Gset^f$ (i.e. the singleton) we get
$$\Gamma_e(M) = \mathrm{Hom}_G(e, M) = M^G$$
So $H^q(e, M) = R^q((\_)^G) = H^q(G, M)$ the usual group cohomology

B.5.2 Hochschild-Serre spectral sequence

If $G \xrightarrow{\tilde{f}} G'$ is a morphism of profinite groups, we have
$$(G'set^f, \tau_C) \xrightarrow{f} (Gset^f, \tau_C)$$
where $f(X)$ is X with the action given by $gx = \tilde{f}(g)x$. It's a morphism of LEX sites, so it induces
$$f_* : G'mod \to Gmod$$
$$M \mapsto \mathrm{Hom}_G(G', M)$$
If $\tilde{f}$ is surjective, i.e. $G' \cong G/N$, we have
$$\mathrm{Hom}_G(G', M) \cong M^N$$
So $R^p f_* M = H^p(N, M)$, so we have the spectral sequence
$$H^p(G/N, H^q(N, M)) \Rightarrow H^{p+q}(G, M)$$
In particular, we have the exact sequence of low degree terms:
$$0 \to H^1(G/N, M^N) \to H^1(G, M) \to (H^1(N, M))^{G/N} \to H^2(G/N, M^N) \to H^2(G, M)$$

B.5.3 The étale site of $Spec(k)$

Theorem B.5.4. *Let k be a field, consider G_k its absolute Galois group. Consider the functor*

$$\gamma : Spec(k)_et \to (G_k - set^f, \tau_C)$$
$$(X = Spec(A)) \mapsto X_{\bar{K}} = Hom_k(A, \bar{k})$$

with the action given by the composition $A \xrightarrow{\alpha} \bar{k} \Rightarrow A \xrightarrow{\alpha} \bar{k} \xrightarrow{\sigma} \bar{k}$. Then γ is an isomorphism of sites (i.e. a bicontinuous equivalence of categories)

Proof. See [Tam12, Âğ2]

B.5.4 Čech, Ãľtale and Galois

Consider for any Ãľtale sheaf F the complex of presheaves $\check{C}^\bullet(F)$ such that $\check{C}^\bullet(F)(U) = \check{C}^\bullet(U, F)$.
If X is quasi-projective over an affine scheme, we have for [Mil16, III.2.17] that $\check{H}^r(U, F) = H^r(U, F)$. In particular,

$$H^r(C^r(F)(U)) = \check{H}^r(U, F) \cong H^r(U, F) = \mathcal{H}^r(F)(U)$$

Hence $H^r(C^\bullet(F)) \cong \mathcal{H}^r(F)$

Proposition B.5.5. *Let X be quasi-projective over an affine scheme, F an Ãľtale sheaf on X. Then*

(a) For every $f : Y \to X$ there is a canonical map $f^*C^\bullet(F) \to C^\bullet(f^*F)$ which is a quasi isomorphism if f is Ãľtale

(b) Let $X = Spec(K)$ and F a sheaf on F corresponding to a G_K-module M. Then $C^\bullet(X, F)$ is the standard resolution of M defined using inhomogeneous chains.

Proof. (a) If $V \to X$ is Ãľtale, there is a canonical morphism defined in B.2

$$\Gamma(V, F) \to \Gamma(V_Y, f^*F)$$

In particular, we have a canonical map

$$\Gamma(U, C^r(F)) \to \Gamma(U_Y, C^r(f^*F)) = \Gamma(U, f_*C^r(f^*F))$$

which by definition commutes with the map induced by the cobords, so we have a canonical morphism of complexes

$$C^\bullet(F) \to f_*C^\bullet(f^*F)$$

which by adjointness gives a canonical map

$$f^*C^\bullet(F) \to C^\bullet(f^*F)$$

And if f is Ãľtale we have

$$H^r(f^*C^\bullet(F)) \cong \mathcal{H}^r(F)_U \cong \mathcal{H}^r(F_U) \cong H^r(C^\bullet(f^*F))$$

(b) If U/X is a finite Galois cover with Galois group G, then $C^\bullet(U/X, F)$ is by definition the standard complex of the G-module $F(U)$ (just checking, see [Mil16, III 2.6]). By passing to the limit we have the result.

□

B.6 The fpqc Site

Definition B.6.1. Consider families of arrows $\{T_i \xrightarrow{f_i} T\}_I$ such that:

1. $T_i \xrightarrow{f_i} T$ is flat for any i and $T = \bigcup_i f_i(T_i)$

2. $\forall\ U \subseteq T$ open affine $\exists$ a finite subset $J \subseteq I$ such that $\exists\ V_j \subseteq T_j,\ j \in J$ open affine such that $U = \bigcup_j f_j(V_j)$

This families give rise to a Grothendieck pretopology called the *fpqc* topology (fidÃĺlemente plate quasi-compact)

One can show that an Ãľtale covering is in fact an fpqc covering (cfr. [Sta], Lemma 33.8.6).
We have this useful lemma:

Lemma B.6.2. *A presheaf F is a sheaf for the fpqc topology if and only if*

1. *It is a sheaf for the Zariski topology*

2. *It satisfies the sheaf property for $\{Spec(B) \twoheadrightarrow Spec(A)\}$ with $A \to B$ faithfully flat*

Proof. cfr [Sta], Lemma 33.8.13 □

So we can now enounce the main theorem:

Theorem B.6.3. *The fpqc topology is subcanonical.*

Proof. We can use the previous lemma: We have that h_X is a Zariski sheaf for the glueing lemma: if we have an open cover U_i of U and arrows $\phi_i : U_i \to X$ such that

$$\phi_{i|U_i \cap U_j} = \phi_{j|U_i \cap U_j}$$

We have that $\exists!\ \phi : U \to X$ such that $\phi_{|U_i} = \phi_i$. In other words,

$$\mathrm{Hom}(U, X) = Eq(\prod(\mathrm{Hom}(U_i, X)) \rightrightarrows \prod(\mathrm{Hom}(U_i \cap U_j, X)) = \prod(\mathrm{Hom}(U_i \times_U U_j, X)))$$

On the other hand, consider $A \to B$ faithfully flat, in particular $\pi : Spec(B) \to Spec(A)$ is surjective. Consider $f : Spec(B) \to X$ such that the following diagram commutes:

$$\begin{tikzcd} & Spec(B \otimes_A B) \arrow[dl] \arrow[dr] & \\ Spec(B) \arrow[dr, "f"] & & Spec(B) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

This means that as a map of sets, f factors through $Spec(A)$, i.e. $\exists\ g$ a map of sets such that the diagram commutes

$$\begin{tikzcd} Spec(B) \arrow[rr, "f"] \arrow[dr, "\pi"] & & X \\ & Spec(A) \arrow[ur, "g"] & \end{tikzcd}$$

Since f is continuous and π is summersive ([Sta], Lemma 28.24.11), g is continuous.
Take now $p \in Spec(A)$ and $g(p) \in U \subseteq X$ for some open affine $U = Spec(R)$. So $p \in g^{-1}(U)$ is open, hence we can choose $a \in A$ such that $p \in D(a) \subseteq g^{-1}(U)$ We have now

$$f_{\pi^{-1}(D(a))} : D(a) \subseteq Spec(B) \to Spec(R)$$

corresponds to a ring map $R \to B[1/a]$. By hypothesis, the following diagram commute:

$$\begin{tikzcd} & Spec((B \otimes_A B)[1/a]) \cong Spec((B[1/a] \otimes_{A[1/a]} B[1/a]) \arrow[dl] \arrow[dr] & \\ Spec(B[1/a]) \arrow[dr, "f_{\pi^{-1}(D(a))}"'] & & Spec(B[1/a]) \arrow[dl, "f_{\pi^{-1}(D(a))}"] \\ & Spec(R) & \end{tikzcd}$$

By definition, $A[1/a] \to B[1/a]$ is faithfully flat and so the following sequence is exact:

$$0 \longrightarrow A[1/a] \longrightarrow B[1/a] \xrightarrow{1\otimes id - id\otimes 1} B[1/a] \otimes_{A[1/a]} B[1/a]$$

So $R \to B[1/a]$ factors uniquely through $A[1/a]$, hence

$$\begin{tikzcd} D(a) \subseteq Spec(B) \arrow[rr, "f_{|D(a)}"] \arrow[dr, "\pi_{D(a)}"] & & Spec(R) \\ & D(a) \subseteq Spec(A) \arrow[ur, "\exists! \psi_a"] & \end{tikzcd}$$

So we get that $\forall\ p \in Spec(A)\ \exists D(a)$ and ψ_a such that the previous triangle commutes, hence the ψ_a glue to a map $Spec(A) \to X$, hence

$$0 \longrightarrow h_X(Spec(A)) \xrightarrow{\pi(\_)} h_X(Spec(B)) \longrightarrow h_X(Spec(B \otimes_A B))$$

is exact, so h_X is a sheaf. □

In particular, h_X is a sheaf for the Ãľtale topology. Moreover, by the same argument, we have that the internal hom: let $\pi : U \to X$ be a map in some site (X, τ) less fine then fpqc, then the presheaf $\mathcal{H}om(F, G)(U) = \mathcal{H}om(\pi^* F, \pi^* G)$ is a sheaf and we have a bifunctor

$$\mathcal{H}om(\_, \_) : Sh_\tau(X)^{op} \times Sh_\tau(X) \to Sh_\tau(X)$$

which is left exact in the two variables, so we can derive it and obtain $\mathcal{E}xt$ (by the same means of Ext we can check that it is the same if we derive the first or the second variable)

B.7 Artin-Schreier

By Yoneda lemma, we have that if h_X is represented by a commutative group scheme, then h_X is a presheaf of abelian groups.

Definition B.7.1. Let X be a scheme. Then the sheaf

$$\mathbb{G}_a := \mathrm{Hom}_{Sch}(\_, \mathbb{G}_a) = \mathrm{Hom}_{Rings}(\mathbb{Z}[T], \_)$$

is a sheaf of abelian groups.

We have a natural inclusion of sites $\epsilon : XZar \hookrightarrow X_{et}$ which induces a left exact functor

$$\epsilon^s Sh(X_{Zar}) \to Sh(X_{et})$$
$$F \mapsto (U \xrightarrow{\pi} X \mapsto \Gamma(U, \pi^* F))$$

Which trivially preserves injectives since π^* does. So if F is a quasi coherent $\mathcal{O}_X$-module it gives a spectral sequence

$$H^p_{Zar}(X, R^q \epsilon^s F) \Rightarrow H^{p+q}_{\acute{e}t}(X, F_{\acute{e}t})$$

where $F_{\acute{e}t}(U \to X) = \Gamma(U, F \otimes_{\mathcal{O}_X} \mathcal{O}_U)$, it is a sheaf for the faithfully flat descent (see [Fu11, Ch. 1] and [Tam12, 3.2.1])

Theorem B.7.2. *If F is a quasicoherent Zariski $\mathcal{O}_X$-module, then $R^q \epsilon^s F = 0$ for $q > 0$, so in particular*

$$H^p_{\acute{e}t}(X, F_{\acute{e}t}) = H^p_{Zar}(X, F)$$

Proof. [Tam12, 4.1.2]

Consider a scheme X of characteristic p, we have the Frobenius morphism

$$Frob : \mathbb{G}_a \to \mathbb{G}_a$$

Theorem B.7.3. *The morphism $\wp = Frob - Id : \mathbb{G}_a \to \mathbb{G}_a$ is epi with kernel $\mathbb{Z}/p\mathbb{Z}$*

Proof. Consider $U \to X$ étale and $s \in Ker(\wp)(U)$, i.e. $s \in \Gamma(U, \mathcal{O}_U)$ such that $s^p = s$. By definition, this are all and only the elements of the image of the characteristic morphism $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \Gamma(U, \mathcal{O}_U)$, hence we have a left exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a$$

The surjectivity comes from the fact that for every ring A of characteristic p the Artin-Schreier algebra $A \hookrightarrow A[T]/(T^p - T - a)$ is free and Ãľtale. In fact, it's enough to show that $\forall\ U \to X\ \exists\ \{U_i \to U\}$ an Ãľtale covering such that $\forall\ s \in \mathcal{O}_U(U)^\times\ \exists\ a_i \in \mathcal{O}_{U_i}(U_i)$ such that $a_i^p - a_i = s_{|U_i}$.
Consider an open affine cover $U = \bigcup_j V_j$ with $V_j = Spec(A_j)$. Hence we have an Ãľtale surjective map

$$A_j[T]/(T^p - T - s_{|V_j}) \longrightarrow A_j$$

So take $U_j = Spec(A_j[T]/(T^p - Ts_{|V_j})$, we have that $U_j \twoheadrightarrow V_j \hookrightarrow U$ is Ãľtale and $\bigcup U_j = \bigcup V_j = X$, so we have an Ãľtale covering $\{U_i \to U\}$ such that

$$s_{U_i} = T^p - T$$

□

So we have an exact sequence (called Artin Schreier exact sequence):

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0$$

So if X has dimension d $H^r(X, \mathbb{G}_a) = H^r_{Zar}(X, \mathcal{O}_X) = 0$ for $r > 2d$, and we have a bounded exact sequence in cohomology

$$0 \to \dots H^r_{ét}(X, \mathbb{Z}/p\mathbb{Z}) \to H^r_{Zar}(X, \mathcal{O}_X) \xrightarrow{\wp} H^r_{Zar}(X, \mathcal{O}_X) \dots \to H^{2d}_{ét}(X, \mathbb{Z}/p\mathbb{Z}) \to 0$$

B.8 Kummer theory

Definition B.8.1. Let X be a scheme. Then the sheaf

$$\mathbb{G}_m := \mathrm{Hom}_{Sch}(\_, \mathbb{G}_m) = \mathrm{Hom}_{Rings}(\mathbb{Z}[T, T^{-1}], \_)$$

is a sheaf of abelian groups.

B.8.1 Useful exact sequences

Kummer Exact Sequence

Consider $n \in \mathbb{Z}$ and the morphism of sheaves given by the n-th power:

$$\mathbb{G}_{m,X} \xrightarrow{(\_)^n} \mathbb{G}_{m,X}$$

It is clear that
$$Ker((\_)^n_X) = \{x \in \mathcal{O}_X(X)^\times : x^n = 1\} = \mu_n$$
So we have a left-exact sequence
$$0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{(\_)^n} \mathbb{G}_{m,X}$$

Proposition B.8.2. *If n is invertible in X, then the sequence is exact*

Proof. It's enough to show that $\forall\ U \to X\ \exists\ \{U_i \to U\}$ an Ãľtale covering such that $\forall\ s \in \mathcal{O}_U(U)^\times\ \exists\ a_i \in \mathcal{O}_{U_i}(U_i)$ such that $a_i^n = s_{|U_i}$.
Consider an open affine cover $U = \bigcup_j V_j$ with $V_j = Spec(A_j)$.By definition, $n \in A_j^\times$ and by the properties of morphisms of rings, $s_{|V_j} \in A_j^\times$, so $ns_{|V_j} \in A_j^\times$. Hence we have an Ãľtale surjective map
$$A_j[T]/(T^n - s_{|V_j}) \longrightarrow A_j$$
So take $U_j = Spec(A_j[T]/(T^n - s_{|V_j}))$, we have that $U_j \twoheadrightarrow V_j \hookrightarrow U$ is Ãľtale and $\bigcup U_j = \bigcup V_j = X$, so we have an Ãľtale covering $\{U_i \to U\}$ such that
$$s_{U_i} = T^n$$
□

Exact sequence for the Zariski topology

Let X be any scheme, recall that a *Prime Weil divisor* is a closed irreducible subscheme of codimension 1.

Definition B.8.3. The sheaf of Weil divisors on X_{Zar} is
$$Div_X(U) = \mathbb{Z}^{Z \text{ prime Weil divisor}} \Rightarrow Div_X = \bigoplus_{codim(Z)=1} i_{Z*}\mathbb{Z}$$

Proposition B.8.4. *If X is regular connected, then we have an exact sequence of Zariski sheaves:*
$$0 \to \mathcal{O}_X^\times \to \mathcal{K}^\times \to Div_X \to 0$$

Proof. If $U \subseteq X$ affine, $U = Spec(A)$, we have the exact sequence
$$0 \to A^\times \to K^\times \to Div(A) = \mathbb{Z}^{\wp: ht(\wp)=1}$$
So we have a left exact sequence
$$0 \to \mathcal{O}_X^\times \to \mathcal{K}^\times \to Div_X$$
And on the stalk,
$$0 \to \mathcal{O}_{X,x}^\times \to \mathcal{K}^\times \to Div(\mathcal{O}_{X,x})$$
And since X is regular, $\mathcal{O}_{X,x}$ is a regular local ring $\Rightarrow$ UFD, so every prime of height 1 is principal for Krull Hauptidealsatz, hence the sequence is exact □

Remark B.8.5. If η is the generic point and $g : \eta \to X$ is the inclusion, then $\mathcal{K}^\times = g_*(K^\times)$

Exact sequence for the Ãľtale topology

Theorem B.8.6. *If X is regular connected, then we have an exact sequence of Ãľtale sheaves:*

$$0 \to \mathbb{G}_{m,X} \to g_*\mathbb{G}_{m,K} \to Div_X \to 0$$

Proof. g is dominant so $\mathbb{G}_{m,X} \to g_*\mathbb{G}_{m,K}$ is injective. Consider $U \to X$ an Ãľtale connected scheme, so U is regular, hence on U_{zar} we have

$$0 \to \mathbb{G}_{m,U} \to K(U)^\times \to Div_U \to 0$$

exact, so we get the exactness on the Ãľtale site.

□

B.8.2 Cohomology of $\mathbb{G}_m$

Recall the isomorphism:

$$Sh_{\mathcal{Ab}}(Spec(k)_{et}) \xrightarrow{\sim} G_kMod$$

$$F \longrightarrow \varinjlim_{H \leq G_k \text{ open}} F(\gamma^{-1}(EG_k/H)) = \varinjlim_{k'/k \text{ finite}} F(Spec(k'))$$

Lemma B.8.7. *Consider $X = X_1 \coprod ... \coprod X_n$, consider the Zariski cover $\{X_i \to X\}$ we have that $\forall$ F sheaf*

$$H^q(X, F) = \prod H^q(X_i, F) \;\forall\; p \geq 0$$

Proof. Since if $i \neq j$ we have $X_i \times_X X_j = \emptyset$ and $X_i \times_X X_i \cong X_i$, the Čech cobordism is just

$$d_n(a)_{i_0...i_n} = \delta_{i_0...i_n} \sum_{k=0}^{n} (-1)^k a$$

hence it is either the zero map if n is odd and the identity if n is even, hence the Čech complex is exact for any presheaf. So in particular

$$\check{H}^p(\{X_i\}, \underline{H}^q F) = 0 \;\forall\; p \geq 1$$

$$\check{H}^0(\{X_i\}, \underline{H}^q F) = \prod \underline{H}^q F(X_i) = \prod H^q(X_i, F)$$

We have the degenerating spectral sequence

$$\check{H}^p(\{X_i\}, \underline{H}^q F) \Rightarrow H^{p+q}(X, F)$$

So $\check{H}^0(X_i, \underline{H}^q F) \cong H^q(X, F)$, hence the thesis. □

Lemma B.8.8. *Let X be a scheme, $x \in X$. Consider $j : Spec(k(x)) \to X$. Then $R^1 j_*(\mathbb{G}_{m,x}) = 0$*

Proof. $R^1 j_*(\mathbb{G}_{m,x})$ is the sheaf associated to

$$X'/X \text{ Ãľtale} \mapsto H^1(X' \times_X Spec(k(x)), \mathbb{G}_{m,x})$$

Since X'/X is Ãľtale, then $X' \times_X Spec(k(x))$ is the spectrum of an Ãľtale $k(x)$-algebra, hence

$$X' \times_X Spec(k(x)) \cong x'_1 \coprod ... \coprod x'_k$$

with $x_j = Spec(K_j)$ and $K_j/k(x)$ finite separable. So

$$H^1(X' \times_X Spec(k(x)), \mathbb{G}_{m,x}) = \prod_{j=1}^{k} H^1(x'_j, \mathbb{G}_{m,x}) = \prod_{j=1}^{k} H^1(G_{K_j}, \overline{K}_j^\times) = 0$$

The last equality is Hilbert 90. So $R^1 j_*(\mathbb{G}_{m,x})$ is the sheaf associated to $0 \Rightarrow R^1 j_*(\mathbb{G}_{m,x}) = 0$ □

Remark B.8.9. Considering Leray Spectral Sequence for j:

$$H^p(X, R^q j_* \mathbb{G}_{m,x}) \Rightarrow H^{p+q}(x, \mathbb{G}_{m,x})$$

Taking the exact sequence of low-degree terms:

$$H^0(X, R^1 j_* \mathbb{G}_{m,x}) \longrightarrow H^2(X, j_* \mathbb{G}_{m,x}) \longrightarrow H^2(x, \mathbb{G}_{m,x})$$

So using the previous lemma, we have a mono $H^2(X, j_* \mathbb{G}_{m,x}) \rightarrowtail H^2(x, \mathbb{G}_{m,x})$

Proposition B.8.10. *Consider $\mathbb{Z}_x$ the skyscraper sheaf $\mathbb{Z}$ with support $\{x\}$, conisder $j : x \to X$, then*

$$H^1(X, j_* \mathbb{Z}_x) = 0$$

Proof. We have again Leray Spectral Sequence

$$H^p(X, R^q j_* \mathbb{Z}_x) \Rightarrow H^{p+q}(x, \mathbb{Z}_x)$$

which gives the exact sequence of low-degree terms

$$0 \longrightarrow H^1(Z, j_* \mathbb{Z}_x) \longrightarrow H^1(x, \mathbb{Z}_x)$$

And $H^1(x, \mathbb{Z}_x) \cong H^1(G_{k(x)}, \mathbb{Z})$, the action on $\mathbb{Z}$ is trivial, so

$$H^1(G_{k(x)}, \mathbb{Z}) = Hom_{cont}(G_{k(x)}, \mathbb{Z})$$

But $G_{k(x)}$ is compact, $\mathbb{Z}$ is discrete and has no finite nonzero subgroups $\Rightarrow Hom_{cont}(G_{k(x)}, \mathbb{Z}) = 0$ □

Recall the definition of the sheaf of Weil divisor:

$$Div_X = \bigoplus_{x \in X^1} \mathbb{Z}_x$$

With X^1 the subset of points of codimension 1, and the Picard group: if X is normal connected, then

$$K^\times \to Div_X(X) \to Pic(X) \to 0$$

Theorem B.8.11. *If X is regular connected, then:*

i $H^1(X, \mathbb{G}_{m,X}) \cong Pic(X)$

ii $H^2(X, \mathbb{G}_{m,X}) \rightarrowtail H^2(G_{K(X)}, \overline{K(X)}^{\times})$

Proof. Consider η the generic point and $g : K(X) \to X$ its inclusion. We have the exact sequence of Ãľtale sheaves:

$$0 \to \mathbb{G}_{m,X} \to g_*\mathbb{G}_{m,\eta} \to Div_X \to 0$$

We have the long exact sequence in cohomology:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathbb{G}_{m,X}) \arrow[r] & H^0(X, g_*\mathbb{G}_{m,\eta}) \arrow[r] & H^0(X, Div_X) \arrow[dll, rounded corners] \\ & H^1(X, \mathbb{G}_{m,X}) \arrow[r] & H^1(X, g_*\mathbb{G}_{m,\eta}) \arrow[r] & H^1(X, Div_X) \arrow[dll, rounded corners] \\ & H^2(X, \mathbb{G}_{m,X}) \arrow[r] & H^2(X, g_*\mathbb{G}_{m,\eta}) & \end{tikzcd}$$

And since:

- $H^0(X, g_*\mathbb{G}_{m,\eta}) = \mathbb{G}_{m,\eta}(K(X)) = K(X)^{\times}$
- $H^1(X, Div_X) \cong \bigoplus_{x \in X^1} H^1(X, i_*\mathbb{Z}_x) = 0$
- $H^1(X, g_*\mathbb{G}_{m,\eta}) = 0$

We have:

i $K(X)^{\times} \longrightarrow Div_X(X) \longrightarrow H^1(X, \mathbb{G}_{m,X}) \longrightarrow 0$

ii $0 \longrightarrow H^2(X, \mathbb{G}_{m,X}) \longrightarrow H^2(X, g_*\mathbb{G}_{m,\eta})$ and from the previous lemma $H^2(X, g_*\mathbb{G}_{m,\eta}) \rightarrowtail H^2(\eta, \mathbb{G}_{m,\eta}) = H^2(G_{K(X)}, \overline{K(X)}^{\times})$

□

Remark B.8.12. If $H^2(G_{K(X)}, \overline{K(X)}^{\times}) = 0$, then $\forall$ n invertible we have that Kummer exact sequence induces in cohomology

$$\begin{tikzcd} 0 \arrow[r] & \mu_n(\mathcal{O}_X(X)) \arrow[r] & \mathcal{O}_X(X)^{\times} \arrow[r, "n"] & \mathcal{O}_X(X)^{\times} \arrow[dll, rounded corners] \\ & H^1(X, \mu_n) \arrow[r] & Pic(X) \arrow[r, "n"] & Pic(X) \arrow[dll, rounded corners] \\ & H^2(X, \mu_n) \arrow[r] & H^2(X, \mathbb{G}_{m,X}) = 0 & \end{tikzcd}$$

B.9 Cohomology of μ_n

Definition B.9.1. A field K is said to be $C1$ if for all n and all nonconstant homogeneous polynomials $f(T_1 \dots T_n)$ with degree $d < n$ there is $(x_1 \dots x_n) \in K^n \setminus 0$ such that $f(x_1 \dots x_n) = 0$

Proposition B.9.2. *If K is $C1$, then $Br(K) = 0$*

Proof. Let D be a K-division algebra of degree r^2, consider $N : D \to K$ the reduced norm. Since $N(x)N(x^{-1}) = 1$ for all $x \in D \setminus \{0\}$, then N has no nonrtivial zeros, but if $e_1 \dots e_{r^2}$ is a K-basis of D then N is a homogeneous polynomial in $K[T_1 ... T_{r^2}]$ of degree r, so $r \le r^2 \Rightarrow r = 1$. □

Theorem B.9.3 (Tsen). *If k is an algebraically closed field and K is an extension of transcendence degree 1, then K is $C1$.*

Proof. [Del, Arcata, 3.2.3]

Theorem B.9.4. *Let k be an algebraically closed field and X/k be a proper smooth curve with genus g. Then we have that*

$$H^r(X, \mu_n) = \begin{cases} \mu_n(k) & \text{if } r = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } r = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Applying remark B.8.12 we have an exact sequence

$$0 \to H^1(X, \mu_n) \to Pic(X) \xrightarrow{n} Pic(X) \to H^2(X, \mu_n) \to 0$$

We can use the exact sequence

$$0 \to Pic^0(X) \to Pic(X) \xrightarrow{deg} \mathbb{Z} \to 0$$

And since $Pic^0(X)$ can be identified with the group of k-rational points of the Jacobian, which is an abelian variety of dimension g, we have that $Pic^0(X) \xrightarrow{n} Pic^0(X)$ is surjective and its kernel is $(\mathbb{Z}/n\mathbb{Z})^{2g}$, hence we have

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{Z}/n\mathbb{Z})^{2g} \arrow[r] \arrow[d] & H^1(X, \mu_n) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Pic^0(X) \arrow[r] \arrow[d, "n"] & Pic(X) \arrow[r, "deg"] \arrow[d, "n"] & \mathbb{Z} \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & Pic^0(X) \arrow[r] \arrow[d] & Pic(X) \arrow[r, "deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & H^2(X, \mu_n) \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

□

B.10 Sheaves of modules

Definition B.10.1. Let Λ be a (non necessarily commutative) ring and $\mathcal{C}$ a site. We can consider the abelian subcategory $Sh(\mathcal{C}, \Lambda)$ of $Sh(\mathcal{C})$ given by the sheaves of Λ-modules. We can consider the *tensor product* of $F, G \in Sh(\mathcal{C}, \Lambda)$ as the sheafification of

$$(F \text{ ''}\otimes\text{'' } G)(X) \mapsto FX \otimes_\Lambda GX$$

So we have a bufunctor

$$\_ \otimes_\Lambda \_ : Sh(\mathcal{C}, \Lambda) \times Sh(\mathcal{C}, \Lambda) \to \Lambda - mod$$

On the other hand, we can consider the sheaf

$$\mathcal{H}om(F, G)(U) = \mathrm{Hom}_{Sh(X,\Lambda)}(F_U, G_U)$$

This is already a sheaf since $\mathrm{Hom}_{Sh(X,\Lambda)}$ is bi-left exact. It comes straightforward that

$$\_ \otimes_\Lambda G \dashv \mathcal{H}om(G, \_)$$

Since $\mathrm{Hom}_{Sh}(F \otimes_\Lambda G, H) = \mathrm{Hom}_{Psh}(F \text{''}\otimes_\Lambda\text{''} G, H) = \mathrm{Hom}_{Psh}(F, \mathcal{H}om(G, H)) = \mathrm{Hom}_{Sh}(F, \mathcal{H}om(G, H)$
So $\_ \otimes_\Lambda F$ is right exact and $\mathcal{H}om(F, _)$ is left exact. We can derive them and obtain $\mathrm{Tor}^i(\_, G)$ and $\mathcal{E}xt^i(G, \_)$
We say that a sheaf of Λ-modules F is *flat* if $\_ \otimes_\Lambda F$ is exact

Proposition B.10.2. *$Sh(X, \Lambda)$ has enough flat objects.*

Proof. If X is the terminal object, consider a covering $\{U_i \xrightarrow{\phi_i} X\} \in Cov(X)$. Consider the sheaf

$$\phi_{i!}(\Lambda)(V) = \begin{cases} \Lambda & \text{if } V \in Cov(U_i) \\ 0 & \text{otherwise} \end{cases}$$

Then by definition $\phi_{i!}(\Lambda)$ is flat. Consider now $F \in Sh(X, \Lambda)$ and take the free resolutions

$$\oplus \phi_{i!}^{I_i}(U_i) \twoheadrightarrow F(U_i)$$

so we have a flat quotient

$$\oplus_i (\phi_{i!}(\Lambda))^{I_i} \twoheadrightarrow F$$

Remark B.10.3. Suppose that F is locally constant $\mathbb{Z}$-constructible (it is enough finitely presented), then if $\bar{x}$ is a point for the topology considered (when this makes sense) we have

$$\mathcal{H}om(F, G)_{\bar{x}} \cong \mathcal{H}om(F_{\bar{x}}, G_{\bar{x}})$$

Then the flat reslution is also locally free, so if $\oplus_i (\phi_{i!}(\Lambda))^{I_i}$ is as before, consider $\bar{x}$, then for all F

$$R\mathcal{H}om(\oplus_i (\phi_{i!}(\Lambda))^{I_i}, F)_{\bar{x}} = R\mathrm{Hom}_\Lambda(\Lambda^J, F_{\bar{x}}) = \mathrm{Hom}_\Lambda(\Lambda^J, F_{\bar{x}})$$

In particular, if M, N and $P \in D^b(X, \Lambda)$ and P is locally constant $\mathbb{Z}$-constructible, the adjunction gives a quasi isomorphism

$$R\mathrm{Hom}(M, R\mathcal{H}om(N, P)) \cong R\mathrm{Hom}(M \otimes^{\mathbb{L}} N, P)$$

This is in general not true, we usually take M (resp. N) to be $\_ \otimes N$-acyclic (resp. $M \otimes \_$-acyclic)

Remark B.10.4. If $f^{-1} : Y \to X$ is a continuous morphism of sites, $F, G \in Sh(X, \Lambda)$, then $f_*(F \otimes_\Lambda G) \in Sh(Y, \Lambda)$ is the sheafification of

$$(V \mapsto F(f^{-1}V) \otimes_\Lambda G(f^{-1}V)^{\#} = f_*F \otimes f_*G$$

B.11 Henselian fields

Let R be a strictly henselian DVR with fraction field K and residue field k. Let $\overline{K}$ be a separable closure of K and let $I = G_K$.

Proposition B.11.1. *For any torsion I-module with torsion prime to $p = char(k)$, there are canonical isomorphisms*

$$H^q(I, M) \cong \begin{cases} M^I & \text{if } q = 0 \\ M_I(-1) & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Where for any torsion abelian group we set

$$A(-1) := Hom(\varprojlim_{(n,p)=1} \mu_n(k), A)$$

Proof. Consider $P \subseteq I$ be the wild ramification subgroup, it is a profinite p-group. Then $(\_)^P$ is exact in the category of torsion modules with torsion prime to p, since every torsion P-module is a filtered colimit of finite P-modules with order prime to p, and every open normal subgroup of P has index a power of p, so

$$H^1(P, M) = \varinjlim_{U \subseteq P} \varinjlim_{M_i} H^1(P/U, M_i^U) = 0$$

Consider the morphism

$$x \mapsto \frac{1}{[P : Stab_P(x)]} \sum_{g \in P/Stab_P(x)} gx : M \to M^P$$

It induces a morphism $M_P \to M^P$ [3]. We can see that the map induced by the quotient $M^P \to M_P$ is the inverse[4]
So Hochschield-Serre degenerates in degree 2 and

$$H^u(I/P, M^P) \cong H^u(I/P, M_P) \cong H^u(I, M)$$

But since I/P is the Galois group of the maximal tamely ramified extension, it is isomorphic to

$$\prod_{\ell \neq p} \widehat{\mathbb{Z}_\ell} = \varinjlim_{(n,p)=1} \mu_n(k)$$

and the result comes from [Fu11, 4.3.9] □

[3] if $x = g'm - m$, then if $g' \notin Stab_P(x)$, $\sum gg'm - \sum gm = 0$, if $g' \in Stab_P(x)$, then $m = g'(x - m) = g'^2 m$. If $p \neq 2$, $m = g'm$ so $\sum gg'm - \sum gm = 0$. If $p = 2$, then $x = g'(m - g'm) = -g'x = -x$ and since 2 does not divide the torsion $x = 0$

[4] Since if $x \in M^P$ $\sum_{g \in P/Stab_P(x)} gx = [P : Stab_P(x)]x$, and since $M^P \to M_P$ is injective it is an isomorphism

B.12 The Étale site of a DVR

The reference for this section is [Maz73].
Let us fix some notation:

- $\mathcal{O}$ will be be a Discrete Valuation Ring (from now on, DVR) with uniformizer θ, quotient field K and residue field k, always assumed to be perfect.
- $\overline{K}$ a fixed algebraic closure of K and v the extension of the valuation to $\overline{K}$, $K_0 \subseteq K$ the maximal unramified extension of K with respect to v, and its residue field $\overline{k}$ will be the algebraic closure of k, $(\_)_v$ the completion with respect to v (recall that $\overline{K}_{\overline{v}} \cong \overline{K_v}$)
- $G_v = Gal(\overline{K}_v/K_v)$ the decomposition subgroup, $I_v = Gal(\overline{K}_v/\overline{K_0}_v)$ the inertia subgroup. Recall that if $\mathcal{O}$ is henselian, then $G_v = G_K$.
- $G_K = Gal(\overline{K}/K)$, $G_k = Gal(\overline{k}/k)$, $S_K = G_K$-mod (equiv. $Sh_{Ab}(Spec(K)_{et})$) and $S_k = G_k$-mod (equiv. $Sh_{Ab}(Spec(k)_{et})$).
- $\alpha : G_k \xrightarrow{\sim} G_v/I_v$
- $\tau = \alpha^* \pi_* : S_K \longrightarrow S_k$, i.e. $\tau M = M^{I_v}$ with G_k-action induced by α. It is left exact.
- $S_{\mathcal{O}}$ be the mapping cylinder of τ.

Recall ([Tam12]) that if $Y \xrightarrow{i} X$ is a closed immersion and $U = X \setminus Y \xrightarrow{j} X$ is the open immersion of the complementary, then let $\mathcal{G}$ be the mapping cylinder of $\tau = i^* j_*$, we have an equivalence of categories

$$Sh_{et}(X) \xrightarrow{\sim} \mathcal{G} \qquad F \mapsto (j^*F, i^*F, i^*F \xrightarrow{i^*\epsilon^j_F} i^*j_*j^*F = \tau j^*F)$$

where ϵ^j is the counit of the adjunction $j^* \dashv j_*$. Hence, we have

$$Spec(k) \xrightarrow{i} Spec(\mathcal{O}), \quad Spec(\mathcal{O}) \setminus Spec(k) = Spec(\mathcal{O}[\frac{1}{\theta}]) = Spec(K)$$

Hence if $\mathcal{O}$ is henselian (i.e. $G_v = G_K$), $S_{\mathcal{O}}$ is equivalent to the Ãľtale site of $Spec(\mathcal{O})$ via the maps given above and the equivalences, in particular:

a. $j^*F = Fj$ since j is an open immersion (hence Ãľtale), so in the equivalence relative to $Spec(K)$ we get
$$j^*F \leftrightarrow \varinjlim_{L/K \text{ finite}} F(L)$$

b. If now $u \to Spec(k)$ is Ãľtale, then $u = \coprod Spec(\ell_i)$ with ℓ_i/k finite separable, hence i^*F is the sheaf associated to
$$U \mapsto \varinjlim_{\substack{U/Spec(\mathcal{O}) \text{ etale} \\ \text{with lift } U \to Spec(k)}} F(U)$$

But we have a terminal object: it's $U = \coprod(Spec(\mathcal{O}_{L_i}))$ with $\mathcal{O}_{L_i}$ is the integral closure of $\mathcal{O}$ in the unramified extension L_i/K induced by ℓ_i, so in the equivalence relative to $Spec(k)$ we get

$$i^*F \leftrightarrow \varinjlim_{\substack{L/K \text{ finite} \\ \text{unramified}}} F(\mathcal{O}_L)$$

So via this equivalence we have

$$\mathbb{G}_{m\mathcal{O}} = \Big(\varinjlim_{\substack{L/K \text{ finite} \\ \text{unramified}}} U(\mathcal{O}_L), \varinjlim_{L/K \text{ finite}} L^*, \varinjlim_{\substack{L/K \text{ finite} \\ \text{unramified}}} i_L \Big) = (U_0, K_0^*, " \subseteq ")$$

with i_L the inclusion $\mathcal{O}_L^* \subseteq L^*$, U_0 the group of units of the integral closure of $\mathcal{O}$ in K_0 seen as a G_k-module, and $\mathbb{G}_{mK}$ as usual $\overline{K}^*$ with G_K action. Then we have an exact sequence

$$0 \to \mathbb{G}_{m\mathcal{O}} = (\overline{K}^*, U_0, " \subseteq ") \to j_*\mathbb{G}_{mK} = (\overline{K}^*, (\overline{K}^*)^{I_v} = K_0^*, id) \to i_*\mathbb{Z} = (0, \mathbb{Z}, 0) \to 0$$

Which follows directly from the discrete valuation

$$0 \to U_0 \to K_0^* \xrightarrow{v} \mathbb{Z} \to 0$$

Remark B.12.1. If $\mathcal{O}$ is henselian and k is finite, then

$$R^q j_*(\mathbb{G}_{mK}) = (0, R^q\tau(\mathbb{G}_{mK}), 0) = (0, H^q(I_v, \overline{K}^*), 0) = 0$$

The last equality follows from local class field theory (see [Ser62, X,7, prop 12])

Lemma B.12.2. *Let $\mathcal{O}$ be a strictly henselian DVR and $X = Spec(\mathcal{O})$, $i : s \to S$ its closed point, $j : \eta \to S$ its generic point, $I_v = Gal(\overline{\eta}/\eta)$. Then for any sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on η we have*

$$(R^q j_* F)_s = \begin{cases} (F_{\overline{\eta}})^I & \text{if } q = 0 \\ (F_{\overline{\eta}})_I(-1) & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since $\mathcal{O}$ is strictly Henselian, the only Ãľtale neighbourhood of s is X itself, hence

$$(Rj_*F)_s = R\Gamma(S, Rj_*F) = R\Gamma(\eta, F)$$

So $(R^q j_* F)_s = H^q(I, F_\eta)$ and the result comes from proposition B.11.1. □

Lemma B.12.3. *Let X be a noetherian scheme of pure dimension 1, $i : x \to X$ a closed point, M a constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X, we have canonically:*

$$R^q i^! M \cong \begin{cases} M(-1) & \text{if } q = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Consider $X_{\bar{x}}$ the strict localization of X in x. Consider the open immersion

$$j : X \setminus \{x\} \to X$$

and its base change

$$\bar{j} : X_{\bar{x}} \times_X X \setminus \{x\} \to X_{\bar{x}}$$

By hypothesis, $X_{\bar{x}}$ is a strictly local trait and $X_{\bar{x}} \times_X X \setminus \{x\}$ is its generic point, so by lemma B.12.2 we have

$$(R^q j_* j^* M)_{\bar{x}} = (R^q \bar{j}_* \bar{j}^* M)_{\bar{x}} = \begin{cases} M & \text{if } q = 0 \\ M(-1) & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

So the canonical morphism

$$M \to j_* j^* M$$

is an isomorphism: this is trivial on $X \setminus x$ and on x it follows from the formula for $q = 0$. Consider the triangle in $D^+(X, \mathbb{Z}/n\mathbb{Z})$

$$i_* R i^! M \to M \to R j_* j^* M \to$$

which gives in cohomology

$$0 \to i_* i^! M \to M \to j_* j^* M \to i_* R^1 i^! M \to 0$$

So $i^! M = 0$ and $R^1 i^! M = 0$ since the middle arrow is an iso and i_* is fully faithful. Continuing in cohomology we have for $q \geq 2$ isomorphisms

$$R^{q-1} j_* j^* M \cong i_* R^q i^! M$$

So the result follows. □

Remark B.12.4. Using the language of derived categories, this translates as

$$R i^! M = M(-1)[2]$$

Lemma B.12.5. *If X is a trait and F is a sheaf on the open point $j : \eta \to X$, M the corresponding G_η-module, then $H^r(X, j_! F) = 0$*

Proof. Consider the exact sequence

$$0 \to j_! F \to R j_* F \to i_* i^* R j_* F \to 0$$

Recall that $i^* j_* F \cong (M^I)_x$ Since $R\Gamma(X, Rj_* F) = R\Gamma(\eta, F) = R\Gamma(G_\eta, M)$ and $R\Gamma(X, i_* i^* R j_* F) = R\Gamma(x, i^* R j_* F) = R\Gamma(G_x, M^I)$, and since $M^{G_\eta} = (M^I)^{G_x}$, we have the long exact sequence

$$H^r(X, j_! F) \to H^r(G_\eta, M) \xrightarrow{\sim} \mathbb{H}^r(\Gamma(G_x, \Gamma(I, M))) \to$$

Hence $H^r(X, j_! F) = 0$ □

Appendix C

Derived categories

C.1 Triangulated categories

Definition C.1.1. A *triangulated category* is an additive category $\mathcal{C}$ together with a *translation functor*, i.e. an automorphism $T : \mathcal{C} \to \mathcal{C}$, and *distinguished triangles*, i.e. sextuples (X, Y, Z, u, v, w) such that X, Y and Z are objects of $\mathcal{C}$ and $u : X \to Y$, $v : Y \to Z$ and $w : Z \to TX$ are morphisms. Abusing notation, a triangle will be usually written

$$\begin{tikzcd} & Z \arrow[dl, "w"'] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

A morphism of triangles is a triple (f,g,h) forming a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

This data must satisfy the axioms:

TR1
- Triangles are closed under isomorphisms,
- For every $u : X \to X$ there exists Z, v and w such that (X, Y, Z, u, v, w) is a triangle,
- $(X, X, 0, id, 0, 0)$ is a triangle

TR2 (X,Y,Z,u,v,w) is a triangle if and only if $(Y, Z, TX, v, w, -Tu)$ is a triangle

TR3 Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w'), and morphisms $f : X \to X'$, $g : Y \to Y'$ commuting with u and u', then there exists an arrow $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

TR4 (The octohedral axiom) Suppose we have the triangles

- $(X, Y, Z', u, j, \cdot)$,

- $(Y, Z, X', v, \cdot, i)$,
- $(X, Z, Y', vu, \cdot, \cdot)$

Then there exist arrows $f : Z' \to Y'$ and $g : Y' \to X'$ such that

$$\begin{tikzcd} & X' \arrow[dl, "(Tj)i"'] & \\ Z' \arrow[rr, "f"] & & Y' \arrow[ul, "g"'] \end{tikzcd}$$

is a triangles and

$$\begin{tikzcd} Z' \arrow[r, "f"] \arrow[d] & Y' \arrow[dl] \\ TX & \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "g"] & X' \\ Z \arrow[u] \arrow[ur] & \end{tikzcd}$$

are commutative

The same definition with reverse arrows leads to a *cotriangulated category*. If $\mathcal{C}$ is triangulated, then $\mathcal{C}^{op}$ is cotriangulated

Definition C.1.2.
- A functor $F : \mathcal{C} \to \mathcal{C}'$ between two triangulated categories is called a *covariant ∂-functor* if it commutes with the translation functor and preserves triangles
- A functor $H : \mathcal{C} \to \mathcal{A}$ from a triangulated category to an abelian category is called a *covariant cohomological functor* if for any triangles (X, Y, Z, u, v, w) the long exact sequence

$$\cdots H(T^iX) \xrightarrow{T^iu} H(T^iY) \xrightarrow{T^iv} H(T^iZ) \xrightarrow{T^iw} H(T^{i+1}X) \xrightarrow{T^{i+1}u} H(T^{i+1}Y) \cdots$$

is exact

We will write $H^i(X)$ for $H(T^iX)$. The same definition applies to contravariant homological functors by reversing the arrows and consider cotriangulated categories.

Proposition C.1.3. *a) The composition of any two morphisms in a triangle is zero*

b) If $\mathcal{C}$ is triangulated and M is an object, then $Hom_{\mathcal{C}}(\_, M)$ and $Hom_{\mathcal{C}}(M, \_)$ are ∂ functors.

c) In the situation of $TR3$, if f and g are isomorphisms then also h is

Proof. a) Let (X, Y, Z, u, v, w) be a triangle. By $TR2$, $(Y, Z, TX, v, w, -Tu)$ is a triangle, so it is enough to show that $uv = 0$. By $TR1$, $(Z, Z, 0, id, 0, 0)$ is a triangle, we have $w : Y \to Z$ and $id : Z \to Z$ satisfying the hypothesis of $TR3$, so there is $h : TX \to 0$ such that $T(v)(-T(u)) = 0$, and since T is an automorphism, it is conservative, hence $uv = 0$

b) Let (X, Y, Z, u, v, w) be a triangle. By $TR2$, it is enough to show that

$$\mathrm{Hom}_C(M, X) \to \mathrm{Hom}_C(M, Y) \to \mathrm{Hom}_C(M, Z)$$

is exact. By a), the composition is zero. So take $g \in \mathrm{Hom}_{\mathcal{C}}(M, Y)$ such that $vg = 0$. Then the triangles $(M, 0, TM, 0, 0, id)$ and $(Y, Z, TX, v, w, -Tu)$, and the arrows $g : M \to Y$ and $0 : 0 \to Z$ satisfy $TR3$, hence we have an arrow $f' : TX \to TM$ such that $-Tuf' = Tg$, and since T is an automorphism we have that $f' = -Tf$, so we have f st $uf = g$.
With the same proof we have $Hom_{\mathcal{C}}(\_, M)$ is a contravariant cohomological functor.

c) Consider the situation in $TR3$ and apply $\mathrm{Hom}_{\mathcal{C}}(Z', \_)$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(Z', X) \arrow[r, "u()"] \arrow[d, "f()"] & \mathrm{Hom}(Z', Y) \arrow[r, "v()"] \arrow[d, "g()"] & \mathrm{Hom}(Z', Z) \arrow[r, "w()"] \arrow[d, "h()"] & \mathrm{Hom}(Z', TX) \arrow[r, "Tu()"] \arrow[d, "Tf()"] & \mathrm{Hom}(Z', TY) \arrow[d, "Tg()"] \\
\mathrm{Hom}(Z', X') \arrow[r, "u'()"] & \mathrm{Hom}(Z', Y') \arrow[r, "v'()"] & \mathrm{Hom}(Z', Z') \arrow[r, "w'()"] & \mathrm{Hom}(Z', TX') \arrow[r, "Tu'()"] & \mathrm{Hom}(Z', TY)
\end{tikzcd}$$

where $f()$, $g()$, $Tf()$ and $Tg()$ are isomorphisms, hence $h()$ is an isomorphism. Take $\phi = (h())^{-1}(id_{Z'}) \in \mathrm{Hom}(Z', Z)$, we have $h\phi = id_Z$. Using now $\mathrm{Hom}(\_, Z)$ we have $\psi \in \mathrm{Hom}(Z, Z')$ such that $\psi h = id_Z$, hence $\phi = \psi = h^{-1}$.

C.1.1 The homotopy category

Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ the homotopy category. Let $f : X \to Y$, then we define the mapping cone $Cone(f) = X[1] \oplus Y$ with differentials given by $\begin{pmatrix} d_X[1] & f[1] \\ 0 & d_Y \end{pmatrix}$. It well-posed since if $f \sim f'$, i.e. $f - f' = s_n d^n + s_{n+1} d^{n+1}$, then $Cone(f) \cong Cone(f')$ as complexes with isomorphism given by $\begin{pmatrix} Id_X & 0 \\ s & Id_Y \end{pmatrix}$, which has inverse $\begin{pmatrix} Id_X & 0 \\ -s & Id_Y \end{pmatrix}$
In particular, $Cone(id_X)$ is null homotopic: consider the maps $\begin{pmatrix} 0 & 0 \\ id_{X_n} & 0 \end{pmatrix} : (X[1])_n \oplus X_n \to (X[1])_{n-1} \oplus X_{n-1}$, they give the homotopy:

$$\begin{pmatrix} -d_X^{n+1} & id_{X_{n+1}} \\ 0 & d_X^n \end{pmatrix}\begin{pmatrix} 0 & 0 \\ id_{X_{n+1}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ id_{X_{n-1}} & 0 \end{pmatrix}\begin{pmatrix} -d_X^n & id_{X_n} \\ 0 & d_X^n \end{pmatrix} = \begin{pmatrix} id_{X_{n+1}} & 0 \\ d_X^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -d_X^n & id_{X_n} \end{pmatrix} = \begin{pmatrix} id_{X_{n+1}} & 0 \\ 0 & id_{X_n} \end{pmatrix}$$

Theorem C.1.4. *$K(\mathcal{A})$ has a structure of triangulated category with translation functor the shifting operator, i.e. $T(X) = X[1]$ and $T^n(X) = X[n]$, and triangles given by sextuples homotopically equivalent to mapping cones, i.e. (X, Y, Z, u, v, w) is a triangle if and only if we have quasi isomorphisms with*

Proof. We have to check the axioms: The first axiom comes by definition and the previous remark shows that $(X, X, 0, id_X, 0, 0)$ is a triangle.
The other axioms come from technical details (see [?, Tag 014P]). $\square$

Corollary C.1.5. *Using the same idea, one can show that $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are full trinagulated subcategories of $K(\mathcal{A})$*

Proof. [?, Tag 014P]

Remark C.1.6. Let $H : K(\mathcal{A}) \to \mathcal{A}$ the functor that sends a complex K into $H^0(K)$ and let H^i be HT^i. It is a cohomological functor.

Proof. Let $u : X \to Y$ be a morphism of complexes. We have an exact sequence

$$0 \to Y \to Cone(u) \to X[1] \to 0$$

so by the snake lemma we have a long exact sequence in cohomology

$$H^i(Y) \to H^i(Cone(u)) \to H^{i+1}(X) \xrightarrow{\delta} H^{i+1}(Y)$$

we need to show that $\delta = u$, but by the construction with the snake lemma, δ is given by the diagram:

$$\begin{tikzcd}
 & & & Z^n(Y) \arrow[d, hook] & \\
0 \arrow[r] & Y^n \arrow[r] \arrow[d] & Cone(u)^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] \arrow[l, dotted, bend right, "(1,0)"'] & 0 \\
0 \arrow[r] & Y^{n+1} \arrow[r] \arrow[d] & Cone(u)^{n+1} \arrow[r] \arrow[l, dotted, bend left, "(0,1)"] & X^{n+2} \arrow[r] & 0 \\
 & B^{n+1}(Y) & & &
\end{tikzcd}$$

So since $\delta = H^n(\delta')$ where δ' given by

$$\delta' = (0,1)\begin{pmatrix} -d_X^{n+1} & u^n \\ 0 & d_Y^n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = u^n$$

hence if (X, Y, Z, u, v, w) is a triangle we have the long exact sequence

$$\cdots H^i(Y) \to H^i(Cone(u)) \xrightarrow{H^n(u)} H^{i+1}(X) \cdots$$

C.2 Localization

Definition C.2.1. Let $\mathcal{C}$ be a category. Then a collection S of arrows is said to be a *multiplicative syuem* if it satisfies the following axioms:

FR1 S is closed under composition and $id_X \in S$ for all X

FR2 If $s, s' \in S$, u, u' any arrows that form the diagrams $\begin{tikzcd} & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "u"] & \cdot \end{tikzcd}$ $\begin{tikzcd} \cdot \arrow[r, "u'"] \arrow[d, "s'"] & \cdot \\ \cdot & \end{tikzcd}$ then

we have $t, t' \in S$ and v, v' any arrows such that we have commutative squares

$$\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "t"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "u"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "u'"] \arrow[d, "s'"] & \cdot \arrow[d, "t'"] \\ \cdot \arrow[r, "v'"] & \cdot \end{tikzcd}$$

FR3 If $f, g : X \to Y$ are any parallel arrows in $\mathcal{C}$, then the following conditions are equivalent:

(i) There exists $s \in S$ such that $sf = sg$

(ii) There exists $t \in S$ such that $ft = gt$

Definition C.2.2. If $\mathcal{C}$ is a category and S is a collection of arrows, then the localization of $\mathcal{C}$ with respect to S is given by a category $\mathcal{C}_S$ and a functor $Q : \mathcal{C} \to \mathcal{C}_S$ such that

a) For all $s \in S$, $Q(s)$ is an isomorphism

b) For every functor $F : \mathcal{C} \to \mathcal{D}$ such that for all $s \in S$, $F(s)$ is an isomorphism, then there is a unique functor $F' : \mathcal{C}_S \to \mathcal{D}$ such that $F = F'Q$

If the localization exists, it is unique up to isomorphisms of categories (standard argument).

Proposition C.2.3. *Let S be a multiplicative system, then the localization $\mathcal{C}_S$ exists and is given by:*

$Ob(\mathcal{C}_S) \quad = Ob((C))$

$Hom_{\mathcal{C}_S}(X, Y) \; = \varinjlim_{I_X^{op}} Hom_{\mathcal{C}}(X', Y)$

where I_X is the category whose objects are $\{(X', s), s \in S, s : X' \to X\}$ and arrows are commutative diagrams

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "s_1"'] & & X_2 \arrow[dl, "s_2"] \\ & X & \end{tikzcd}$$

Furthermore, if $\mathcal{C}$ is additive, so it $\mathcal{C}_S$.

Proof. We have that

(i) $I_X \neq \emptyset$ since $(X, id_X) \in I_X$

(ii) $(X_1, s_1), (X_2, s_2)$, consider (W, t_1, t_2) as in $FR2$, we have $u = s_1 t_1 = s_2 t_2 \in S$, so $(W, u) \in I_X$ and by $FR2$ we have that t_1, t_2 induces morphisms

$$\begin{tikzcd} & (X_1, s_1) \\ (W, u) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & \\ & (X_2, s_2) \end{tikzcd}$$

so I_X^{op} is filtered. So $f \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ is represented by a diagram

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$$

with $s \in S$, and two diagrams (X', s, a) and (X'', t, b) define the same morphism if there exius $u : \widetilde{X} \to X$ in S and a diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[ddl, bend left, "u"] & \\ X' \arrow[d, "s"'] \arrow[drr] & & X'' \arrow[d, "b"] \arrow[dll] \\ X & & Y \end{tikzcd}$$

To compose morphisms

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y' \arrow[dl, "t"'] \arrow[dr, "b"] & \\ Y & & Z \end{tikzcd}$$

one uses $FR2$ to find a commutative diagram

$$\begin{tikzcd} & & X'' \arrow[dl, "t'"'] \arrow[dr, "c"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "a"] & & Y' \arrow[dl, "t"'] \arrow[dr, "b"] & \\ X & & Y & & Z \end{tikzcd}$$

By the same argument as before, we can see that the composition does not depend on X', Y' and X'', so $\mathcal{C}_S$ is well defined, and by construction

$$Q : \mathcal{C} \to \mathcal{C}_S$$

such that if $X \xrightarrow{f} Y$, then $Q(f)$ is given by the diagram

$$\begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

If now $s \in S$, then $Q(s)$ is an isomorphism with inverse

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "id_Y"] & \\ X & & Y \end{tikzcd}$$

If $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism, we can define F' as

$$F'(f) = \varinjlim_{I_X} F(s)^{-1} F(a)$$

F' is unique by definition and $F = F'Q$ by definition.
Moreover, if $\mathcal{C}$ is additive, since I_X^{op} is filtered then

$$\varinjlim_{I_X} \mathrm{Hom}_{\mathcal{C}}(X', Y)$$

is an abelian group (filtered colimits exists in Ab) and the composition distributes. □

Definition C.2.4. Let $\mathcal{C}$ be a triangulated category and S a multiplicative system. S is said to be compatible with the triangulation if

FR4 $s \in S$ if and only if $Ts \in S$

FR5 As in $TR3$, if $f, g \in S$ then $h \in S$

Proposition C.2.5. *If $\mathcal{C}$ be a triangulated category and S a multiplicative system compatible with the triangulation, then C_S has a unique structure of triangulated category such that Q is a ∂-functor universal for all δ-functors, i.e. such that if $F : \mathcal{C} \to \mathcal{D}$ is a ∂-functor between triangulated categories such that for all $s \in S$ $F(s)$ is an isomorphism, then there exists a unique ∂-functor $F' : \mathcal{C}_S \to \mathcal{D}$ such that $F = F'Q$.*

Proof. Easy but technical, see [Sta, Tag 05R6]

Proposition C.2.6. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ a full subcategory, let S be a multiplicative system in $\mathcal{C}$ such that $\mathcal{D} \cap S$ is a multiplicative system in $\mathcal{D}$. Assume that one of the following condition is true:*

(i) For every morphism $s : X' \to X$ with $s \in S$ and $X \in \mathcal{D}$, there is a morphism $f : X'' \to X'$ such that $X'' \in \mathcal{D}$ and $sf \in S$

(ii) For every morphism $s : X \to X'$ with $s \in S$ and $X \in \mathcal{D}$, there is a morphism $f : X' \to X''$ such that $X'' \in \mathcal{D}$ and $fs \in S$ (the dual statement)

Then the natural functor $\mathcal{D}_{S \cap D} \to \mathcal{C}_S$ is fully faithful

Proof. Straightforward by definition of $\mathrm{Hom}_{\mathcal{D}}(X, Y)$

Proposition C.2.7. *Let $\mathcal{C}$ be a category, S be a multiplicative system and $Q : \mathcal{C} \to \mathcal{C}_S$ the localization. Let $\mathcal{D}$ be a category and $F, G : \mathcal{C}_S \to \mathcal{D}$ two functors. Then the natural map*

$$\alpha : Nat(F, G) \to Nat(FQ, GQ)$$

is an isomorphism

Proof. Since $Ob(\mathcal{C}) = Ob(\mathcal{C}_{\mathcal{S}})$, α is injective. Since every morphism in $\mathcal{C}_S$ is represented by a diagram

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$$

and since Fs and Gs are isomorphisms, we have that if $\eta : FQ \to GQ$, then η'_X is given by the composition:

$$\begin{tikzcd}
 & FX' \arrow[dl, "Fs"'] \arrow[d, "\eta_X"] \arrow[dr, "Fa"] & \\
FX \arrow[d, dotted, "Gs\eta_X(Fs)^{-1}"'] & GX' \arrow[dl, "Gs"'] \arrow[dr, "Ga"] & FY \arrow[d, "\eta_Y"] \\
GX & & GY
\end{tikzcd}$$

C.3 The definition of Derived Category

Proposition C.3.1. *Let $\mathcal{C}$ be a triangulated category, $\mathcal{A}$ an abelian category and $H : \mathcal{C} \to \mathcal{A}$ a cohomological functor. Consider*

$$S := \left\{ s \in Arr(\mathcal{C}) \text{ such that } H(T^i s) \text{ is an isomorphism for all } i \in \mathbb{Z} \right\}$$

Then S is a multiplicative system compatible with the triangulation.

Proof. FR1 Trivial by definition

FR2 Let $s \in S$ and a diagram

$$\begin{tikzcd}
 & Z \arrow[d, "s"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}$$

Using $TR1$, complete s to a triangle (Z, Y, N, s, f, g). Complete fu to a triangle (W, X, N, t, fu, h). Then we have a commutative square

$$\begin{tikzcd}
X \arrow[r, "fu"] \arrow[d, "u"] & N \arrow[d, "id_N"] \\
Y \arrow[r, "f"] & N
\end{tikzcd}$$

so by $TR3$ there is a map $v : W \to Z$ giving a morphism of triangles

$$\begin{tikzcd}
W \arrow[r, "t"] \arrow[d, "v"] & X \arrow[r, "fu"] \arrow[d, "u"] & N \arrow[r, "h"] \arrow[d, "id_N"] & TW \arrow[d, "Tv"] \\
Z \arrow[r, "s"] & Y \arrow[r, "f"] & N \arrow[r, "g"] & TZ
\end{tikzcd}$$

Then $sv = ut$, so it remains to prove that $t \in S$.
Since $s \in S$, we have the long exact sequence

$$HT^iZ \xrightarrow{\sim} HT^iY \to HT^iN \to HT^{i+1}Z \xrightarrow{\sim} HT^{i+1}Y$$

hence $HT^iN = 0$ for all $i \in \mathbb{Z}$. Hence the long exact sequence

$$HT^{i-1}N = 0 \to HT^iW \xrightarrow{HT^it} HT^iX \to HT^iN = 0$$

shows that HT^it is an isomorphism for all i, hence $t \in S$. The dual statement is analogous.

FR3 Let $f : X \to Y$ be a morphism. Since $\mathcal{C}$ is additive, it is enough to show the equivalence of

(i′) There exists $s : Y \to Y'$, $s \in S$ such that $sf = 0$

(ii′) There exists $t : X' \to X$, $t \in S$ such that $ft = 0$

Suppose (i') holds, so complete s into a triangle (Z, Y, Y', v, s, u). Since $sf = 0$ and

$$\mathrm{Hom}(X, Z) \xrightarrow{v()} \mathrm{Hom}(X, Y) \xrightarrow{s()} \mathrm{Hom}(X, Y')$$

is exact, there exists $g : X \to Z$ such that $vg = f$, so we can again complete g to a triangle (X', X, Z, t, g, w). since now

$$\mathrm{Hom}(X, Y) \xrightarrow{()g} \mathrm{Hom}(X, Z) \xrightarrow{()t} \mathrm{Hom}(X', Z)$$

is exact and $f = vg$, we have $ft = 0$. By the same method as $FR2$, since $s \in S$ we have the long exact sequence

$$HT^iY \xrightarrow{\sim} HT^iY' \to HT^{i+1}Z \to HT^{i+1}Y \xrightarrow{\sim} HT^{i+1}Y'$$

So $HT^iZ = 0$, hence the long exact sequence

$$HT^{i-1}Z = 0 \to HT^iX' \xrightarrow{HT^it} HT^iX \to HT^iZ = 0$$

shows that HT^it is an isomorphism for all i, hence $t \in S$. The other implication is analogous.

FR4 Trivial by definition since T is an automorphism of $\mathcal{C}$

FR5 We have a morphism of long exact sequence

$$\begin{tikzcd}
HT^iX \arrow[r] \arrow[d, "HT^if"] & HT^iY \arrow[r] \arrow[d, "HT^ig"] & HT^iZ \arrow[r] \arrow[d, "HT^ih"] & HT^{i+1}X \arrow[r] \arrow[d, "HT^{i+1}f"] & HT^{i+1}Y \arrow[d, "HT^{i+1}g"] \\
HT^iX' \arrow[r] & HT^iY' \arrow[r] & HT^iZ' \arrow[r] & HT^{i+1}X' \arrow[r] & HT^{i+1}Y'
\end{tikzcd}$$

Since by hypothesis HT^if and HT^ig are isomorphisms, by five lemma HT^ih is an isomorphism, so $h \in S$.

□

Corollary C.3.2. *Let $\mathcal{A}$ be abelian, $K(\mathcal{A})$ the homotopy category, then if Qis is the class of the quasi isomorphisms, is a multiplicative system compatible with the triangulation*

Definition C.3.3. Let $\mathcal{A}$ be abelian. The *derived category* $D(\mathcal{A})$ of $\mathcal{A}$ is defined as $K(\mathcal{A})_{Qis}$. Similarly, we define $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$, and they are all full subcategories via proposition C.2.6

Remark C.3.4. The functor "complex in degree 0" $\mathcal{A} \to D(\mathcal{A})$ is fully faithful and its essential image consists of the complexes such that $H^i(X) = 0$ for $i \neq 0$.

Proof. Let $f : A \to B$ in $\mathcal{A}$. Then $f = 0$ in $D(\mathcal{A})$ if and only if there is a quasi isomorphism $s : B \to X$ such that sf is null homotopic. Since B is in degree zero, $s^i = 0$ for all $i \neq 0$ and by the commutativity of the squares, $s^0 : B \to Ker(d^0_X)$, and since $H^0(B) = B$, s^0 induces an isomorphism

$$s : B \xrightarrow{\sim} H^0(X)$$

so its inverse induces a map

$$t : Ker(d^0_X) \to B$$

such that $td^{-1}_X = 0$ and $ts^0 = id_B$. So if $s^0 f = d^{-1}h$, we have that

$$f = ts^0 f = td^{-1}h = 0$$

So the functor is faithful.
Take now $f \in \mathrm{Hom}_{D(\mathcal{A})}(A, B)$, it is represented by:

$$\begin{tikzcd}
0 \arrow[rr] & & A \arrow[rr] & & 0 & \\
& X^{-1} \arrow[ul] \arrow[dl] \arrow[rr, "d^{-1}"] & & X^0 \arrow[ul, "s"'] \arrow[dl, "a"] \arrow[rr, "d^0"] & & X^1 \arrow[ul] \arrow[dl] \\
0 \arrow[rr] & & B \arrow[rr] & & 0 &
\end{tikzcd}$$

Since $sd^{-1} = 0$ and $ad^{-1} = 0$, we have that s and a factorize through $X^0/Im(d^{-1})$, so we have a quasi isomorphism t which is the identity in degree $\neq 0$ and the passage to the quotient in degree zero. Hence f is also represented by

$$\begin{tikzcd}
0 \arrow[rr] & & A \arrow[rr] & & 0 & \\
& X^{-1} \arrow[ul] \arrow[dl] \arrow[rr, "0"] & & X^0/Im(d^{-1}) \arrow[ul, "s"'] \arrow[dl, "a"] \arrow[rr, "d^0"] & & X^1 \arrow[ul] \arrow[dl] \\
0 \arrow[rr] & & B \arrow[rr] & & 0 &
\end{tikzcd}$$

So taking t the inverse of s in cohomology, we have that if $\iota : Ker(d^0)/Im(d^{-1}) \hookrightarrow X^0/Im(d^{-1})$ is the inclusion, $s\iota t = id$ So f is the image of a map $A \to B$ via the composition with ιt. So the functor is full.
Then we conclude by construction.

C.3.1 Enough injectives

We will see now another description of $D^+(\mathcal{A})$ if $\mathcal{A}$ has enough injectives. We first need three technical lemmas: Let fix an abelian category $\mathcal{A}$

Lemma C.3.5. *Let $f : Z \to I$ be a morphism of complexes of an abelian category such that Z is acyclic, I^n is injective for all n and I is bounded below. Then f is null-homotopic.*

Proof. We will construct an homotopy by induction. For $n << 0$, $I^n = 0$. Then let h^n be zero (since f^n is zero). So supposed that for all $n < p$ we have constructed h_n such that $f^n = h^n d_Z^n + d_I^{n-1} h^{n-1}$. Then consider $g^p := f_p - d_I^{p-1} h^{p-1}$: we have

$$g^p d_Z^{n-1} = d_I^p f^{p-1} - d_I^{p-1}(f^{p-1} - d_I^{p-2} h^{p-2}) = 0$$

Hence g factorizes through $Z^p Im(d_Z^{p-1}) = Z^p/Ker(d^p)$ since Z is acyclic. But since I^p is injective and $Z^p/Ker(d^p) \to Z^{p+1}$ is mono, we have an extension $h^p := Z^{p+1} \to I^p$ such that $h^p d_Z^p = g$, hence $f^p = h^p d_Z^p + d_I^{p-1} h^{p-1}$

Lemma C.3.6. *Let $s : I^\bullet \to Y^\bullet$ a quasi isomorphism of complexes where I^p is injective and $I^\bullet$ is bounded below. Then s is an homotopical equivalence*

Proof. Consider the mapping cone $Z^\bullet = TI^\bullet \oplus Y^\bullet$, then $Z^\bullet$ is acyclic by the long exact sequence in cohomology. Hence the map $v : Z^\bullet \to TI^\bullet$ is null-homotopic by lemma C.3.5. So consider the homotopy

$$(k, t) : TI^\bullet \oplus Y^\bullet \to I^\bullet$$

Then we have that:

$$v = (id_I, 0) = (k, t)d_Z + d_I(k, t) \Rightarrow \begin{cases} id_I = kd_Z + ts + d_I k, \\ 0 = td_Z + d_I t \end{cases}$$

The second one gives that t is a morphism of complexes, the first one that $ts \sim id_I$.

Lemma C.3.7. *1) Let P be a subset of $Ob(\mathcal{A})$ and assume*

(i) Every object of $\mathcal{A}$ admits an injection into an element of P

Then every complex $X^\bullet$ of $K(\mathcal{A})$ admits a quasi isomorphism into a bounded below complex $I^\bullet$ of objects of P such that every map $X^p \to I^p$ is mono.

2) Assume furthermore that P satisfies

(ii) If $0 \to X \to Y \to X \to 0$ is a short exact sequence such that $X \in P$, then $Y \in P$ if and only if $Z \in P$

(iiii) There exists a positive integer n such that if

$$X^0 \to \cdots X^n \to 0$$

is exact and $X^0 \cdots X^{n-1} \in P$, then $X^n \in P$

Then every complex $X \in K(\mathcal{A})$ admits a quasi isomorphism into a complex $I^\bullet$ of objects of P

Proof. 1) We may assume $X^p = 0$ for $p < 0$. Then consider an embedding $X^0 \to I^0$ with $I^0 \in P$. Then we can suppose that we have $I^0 \cdots I^{p-1}$. Consider the pushout

$$\begin{array}{ccc} X^{p-1} & \longrightarrow & X^p \\ \downarrow & & \downarrow \\ I^p/im(I^{p-1}) & \longrightarrow & Q \end{array}$$

Consider an embedding $Q \to I^p$ and define the maps as in the pushout. By construction, $I^\bullet$ is a complex and $X^\bullet \to I^\bullet$ is a quasi isomorphism and every map $X^p \to I^p$ is mono

2) Let i_0 be an integer, and consider the truncated complex

$$0 \to Ker(d^{i_0}) \to X^{i_0} \to \cdots$$

Then by 1) we have a quasi isomorphism into a complex $I^\bullet$ with elements in P with each $X^p \to I^p$ mono. So consider the complex $X_0^\bullet$ as

$$\cdots X^{i_0-2} \to X^{i_0-1} \to I^{i_0} \to I^{i_0+1} \to \cdots$$

Then we have a quasi-isomorphism $X^\bullet \to X_0^\bullet$ such that every map $X^p \to X_0^p$ is mono.
Suppose now that $i_1 \in \mathbb{Z}$ and that we have $X_1^\bullet$ a complex where $X^p \in P$ for $p > i_1$. Take $i_2 < i_1$. We can find by the previous step a quasi isomorphism $X_1^\bullet \to X'^\bullet$ such that $X'^p \in P$ for $p \geq i_2$ and such that every map $X_1^p \to X'^p$ is mono. Then take $Y^p = coker(X_1^p \to X'^p)$, Y^p is an acyclic complex and by property (ii) for $p \geq i_2$ $Y^p \in P$, and for property (iii), for $p \geq i_1 + n$ we have $B^p(Y^\bullet) \in P$ (just take the exact sequence $Y^{p-n} \to \cdots \to Y^p \to B^p \to 0$) Then we have an exact sequence

$$0 \to X_1^i \to Q \to B^i(Y) \to 0$$

where Q is the pushout of the diagram

$$\begin{array}{ccc} X_1^{i-1} & \longrightarrow & X_1^i \\ \downarrow & & \downarrow \\ B^i(X') & \longrightarrow & Q \end{array}$$

So we can define

$$X_2^p = \begin{cases} X'^i & \text{if } p < i_1 + n \\ Q & \text{if } p = i_1 + n \\ X_1^i & \text{if } p > i_1 + n \end{cases}$$

Then by construction $X_1 \to X_2$ is a quasi isomorphism and $X_2^p \in P$ for $p \geq i_2$ and $X_2^p = X_1^p$ for $p > i_1 + n$.
So now if $i_0 > i_1 > \cdots$ is a strictly decreasing sequence of integers, choose X_0 as

in the first step and $X_1, X_2 \cdots$ for $i_1, i_2 \cdots$ as in the second step. Then we have quasi isomorphisms

$$X \to X_0 \to X_1 \to X_2 \cdots$$

and for each p we have that

$$X^p \to X_0^p \to X_1^p \to X_2^p \cdots$$

is eventually constant and eventually in P, hence $\varinjlim X_r$ is the required complex.

□

Proposition C.3.8. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{I}$ be the additive subcategory of injective objects. Then the natural functor*

$$\alpha^+ : K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is fully faithful. If $\mathcal{A}$ has enough injectives, then α^+ is an equivalence.

Proof. We have that $K^+(\mathcal{I})\mathcal{Q}is$ is a multiplicative system in $K^+(\mathcal{I})$ and lemma C.3.6 gives the condition (ii) of proposition C.2.6, hence the natural functor

$$D^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is fully faithful, and lemma C.3.6 says that every quasi isomorphism in $K^+(\mathcal{I})$ is an isomorphism, hence $K^+(\mathcal{I}) = D^+(\mathcal{I})$.
If now $\mathcal{A}$ has enough injectives, apply lemma C.3.7 to $P = \mathcal{I}$ and we have that every object in $D^+(\mathcal{A})$ is isomorphic to one in $K^+(\mathcal{I})$. □

Remark C.3.9. With the dual construction, we can show that if $\mathcal{P}$ is the additive subcategory of injective objects, then the natural functor

$$\alpha^- : K^-(\mathcal{P}) \to D^-(\mathcal{P})$$

is fully faithful. If $\mathcal{A}$ has enough projectives, then α^- is an equivalence.

C.4 Derived Functors

Definition C.4.1. Let $K^*(\mathcal{A})$ be a triangulated subcategory of $\mathcal{A}$. Then $K^*(\mathcal{A}) \cap Qis$ is a multiplicative system in $K^*(\mathcal{A})$. We say that $K^*(\mathcal{A})$ is a *localizing subcategory* if the natural functor

$$K^*(\mathcal{A})_{K^*(\mathcal{A})\cap Qis} \to D(\mathcal{A})$$

is fully faithful and we will denote $D^*(A) := K^*(\mathcal{A}) \cap Qis$

Example C.4.2. $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are localizing subcategories for proposition C.2.6

Definition C.4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $K^*(\mathcal{A})$ a localizing subcategory of $K(\mathcal{A})$, and let

$$F : K^*(\mathcal{A}) \to K(\mathcal{B})$$

be a ∂-functor. Let $Q^* : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ and $Q : K(\mathcal{B}) \to D(\mathcal{B})$ be the localization functors. Then the *right derived functor* of F is a ∂-functor

$$R^*F : D^*(\mathcal{A}) \to D(\mathcal{B})$$

together with a natural transformation of functors from $K^*(\mathcal{A})$ to $D(\mathcal{B})$:

$$\xi : QF \stackrel{\cdot}{\to} R^*FQ^*$$

with the universal property that for every δ-functor

$$G : D^*(\mathcal{A}) \to D(\mathcal{B})$$

and every natural transformation

$$\zeta : QF \stackrel{\cdot}{\to} GQ^*$$

there is a unique natural transformation $\eta : R^*F \to G$ such that the following diagram commutes

$$\begin{tikzcd} QF \arrow[rr, "\zeta"] \arrow[dr, "\xi"'] & & GQ^* \\ & R^*FQ^* \arrow[ur, "\eta Q^*"'] & \end{tikzcd}$$

By the usual argument, if R^*F exists it is unique up to natural isomorphism.

Notation. If $K^*(\mathcal{A})$ is resp. $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ or $K^b(\mathcal{A})$, we will write R^+F, R^-F or R^bF. If there is no confusion, we will simply write RF. We will also write R^pF for $H^p(RF)$

Remark C.4.4.

- If $\phi : F \stackrel{\cdot}{\to} G$ is a natural transformation and both RF and RG exist, then there is a unique $R\phi : RF \to RG$ compatible with ξ. This follows from the definition:

  $$\begin{tikzcd} QF \arrow[r, "\xi_F"] \arrow[d, "Q\phi"] & RFQ^* \arrow[d, dotted, "R\phi Q^*"] \\ QG \arrow[r, "\xi_G"] & RGQ^* \end{tikzcd}$$

  So there is a unique $R\phi$ such that $\xi_G Q\phi = R\phi Q^* \xi_F$

- If $K^{**}(\mathcal{A}) \subseteq K^*(\mathcal{A})$ are two localizing subcategories, then if

  $$F : K^*(\mathcal{A}) \to K(\mathcal{B})$$

  is a ∂-functor, then if Q^{**} is the localization functor for $K^{**}(\mathcal{A})$ we have a map (the symbol "|" indicates the restriction of the functor to the subcategory)

  $$QF_{|K^{**}(\mathcal{A})} = (QF)_{|K^{**}(\mathcal{A})} \to (RFQ^*)_{|K^{**}(\mathcal{A})} = (RF)_{|D^{**}(\mathcal{A})}Q^{**}$$

hence by the unversal property we have

$$R^{**}(F_{|K^{**}(\mathcal{A})}) \to (R^{*}F)_{|D^{**}(\mathcal{A})}$$

In general, it is not an isomorphism, but for all the application we need it will be.

Theorem C.4.5 (Existence). *Let $\mathcal{A}$, $\mathcal{B}$, $K^*(\mathcal{A})$ and F as before, suppose that there is a triangulated subcategory $L \subseteq K^(\mathcal{A})$ such that*

EX1 Every object of $K^*(\mathcal{A})$ admits a quasi isomorphism to an object of L

EX2 If $I^\bullet \in L$ is acyclic, then $FI^\bullet$ is acyclic

Then F admits a right derived functor (RF, ξ) and for every object $I^\bullet \in L$,

$$\xi_{I^\bullet} QF(I^\bullet) \to RFQ^*(I^\bullet)$$

is an isomorphism in $D(\mathcal{B})$.

Proof. First, we need to show that $F_{|L}$ preserves quasi-isomorphisms: let $I_1 \xrightarrow{s} I_2$ a quasi isomorphism, complete it to a triangle $(I_1, I_2, J, s, \cdot, \cdot)$. Then since L is triangulated $J \in L$ and we have already observed that J is acyclic. Hence FJ is acyclic, and since F is a ∂-functor, $(FI_1, FI_2, FJ, Fs, F\cdot, F\cdot)$ is a triangle in $K(\mathcal{B})$, and for the long exact sequence we have that $FI_1 \xrightarrow{Fs} FI_2$ is a quasi isomorphism.
So by the universal property of the localization F induces a functor

$$\overline{F} : L_{Qis} \to D(B)$$

such that $QF = \overline{F}Q_L$.
By hypothesis, L, Qis and $K^*(\mathcal{A})$ satisfy the same hypothesis as proposition C.3.8, hence the full inclusion $T : L_{Qis} \to D^*(\mathcal{A})$ is an equivalence of categories. So fix a quasi inverse

$$U : D^*(\mathcal{A}) \to L_{Qis}$$

and the natural isomorphisms

$$\alpha : 1_{L_{Qis}} \Rightarrow UT \quad \beta : 1_{D^*(\mathcal{A})} \to TU$$

Then we can define $R^*F := \overline{F}U$. We need to define ξ:
Let $X \in K^*(\mathcal{A})$ and let $I \in L$ such that $Q_L(I) = UQ^*(X)$. Then we have an iso in $D^*(\mathcal{A})$:

$$\beta_{Q^*X} : Q^*X \xrightarrow{\sim} TU(Q^*X) = TQ_L(I)$$

Since T is an inclusion, the isomorphism is represented by a diagram

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & I \end{tikzcd}$$

where s, t are quasi isomorphisms, and by hypothesis $EX1$ we can suppose $Y \in L$. Hence applying F we have that Fs is a quasi isomorphisms, so this gives an morphism in $D(\mathcal{B})$

$$\xi_X : QFX \to QFI = \overline{F}Q_L I = \overline{F}UQ^*X = RFQ^*X$$

It is obvious that ξ_X does not depend on Y and it is natural in X. By construction then (RF, ξ) is the derived functor of F (it is constructed to have the universal property). Moreover, if in the construction $X \in L$, we have that $F(t)$ is a quasi-iso, so ξ_X is an iso in $D(\mathcal{B})$. □

Proposition C.4.6. *Let $\mathcal{A}$, $\mathcal{B}$, $K^*(\mathcal{A})$ and F as before, $K^{**}(\mathcal{A}) \subseteq K^*(\mathcal{A})$ another localizing subcategory and that there is a triangulated subcategory $L \subseteq K^(\mathcal{A})$ satisfying the hypothesis of theorem C.4.5 and such that $L \cap K^{**}(\mathcal{A})$ satisfies the hypothesis $EX1$ for $K^{**}(\mathcal{A})$ [1]. Then the natural map*

$$R^{**}(F_{|K^{**}(\mathcal{A})} \to (R^*F)_{|D^{**}(\mathcal{A})}$$

is an isomorphism

Proof. Since if $X \in D^{**}(\mathcal{A})$ is isomorphic to one coming from L, we can suppose $X = QI$ with $I \in L$. By theorem C.4.5, ξ_X is an isomorphism, then by the construction of remark C.4.4 the natural map is an isomorphism.

Corollary C.4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let*

$$F : K^+(\mathcal{A}) \to K(\mathcal{B})$$

be a δ-functor. Then R^+F exists.

Proof. Let $L \subseteq$ be the triangulated subcategory of injective objects. Then by lemma C.3.7, every object of $K^+(A)$ is quasi isomorphic to an object in L, hence $EX1$ is satisfied. Moreover, every quasi isomorphism in L is an isomorphism in $K^+(\mathcal{A})$ by lemma C.3.6, so F preserves quasi isomorphisms, hence F sends acyclic complexes into acyclic complexes. So the hypotheses of theorem C.4.5 are satisfied.

Corollary C.4.8. *Let $\mathcal{A}$, $\mathcal{B}$ abelian categories and F an additive functor. Assume there is $P \subseteq Ob(\mathcal{A})$ satisfying hypotheses (i) and (ii) of lemma C.3.7 and also*

(iv) F preserves short exact sequences of objects of P

Then denoting again by F the induced δ-functor $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$, R^+F exists.

Proof. Let L be the subcategory of $K^+(\mathcal{A})$ made of object of P. Since (ii) holds, P is closed for direct sums, hence L is closed for mapping cones, so it is triangulated. Again, for lemma C.3.7, $EX1$ is satisfied.

[1] hence it satisfies the hypothesis of theorem C.4.5, since $EX2$ comes from $K^*(\mathcal{A})$, i.e. both R^*F and $R^{**}(F_{|K^{**}(\mathcal{A})}$ exist

Suppose $I^\bullet$ acyclic: since it is bounded below we have $Ker(d^n) = 0$ for $n << 0$, hence for (ii) $Ker(d^n) \in P$ for $n << 0$. If now $Ker(d^n) \in P$, we have the exact sequence

$$0 \to Ker(d^n) \to I^n \to Ker(d^{n+1}) \to 0$$

So since $Ker(d^n \in P)$ and $I^n \in P$ by (ii) $Ker(d^{n+1}) \in P$, hence for all n $Ker(d^n) \in P$. So since F preserves exact sequences

$$0 \to F(ker(d^n)) \to F(I^n) \xrightarrow{F(d^n)} F(Im(d^n)) \to 0$$

is exact, hence $Im(F(d^n)) = F(Im(d^n)) = F(ker(d^{n+1}) = ker(F(d^{n+1}))$.

□

Corollary C.4.9. *If F, $\mathcal{A}$, $\mathcal{B}$ are as in corollary C.4.8 and F has finite cohomological dimension[2] RF exists and its restriction to $D^+(\mathcal{A})$ is equal to R^+F*

Proof. Consider P' to be the collection of all F-acyclic objects and $L' \subseteq K(\mathcal{A})$ be the triangulated subcategory made of objects in P'. So (iv) holds by definition. Then, since $P \subseteq P'$, hypothesis (i) of lemma C.3.7 is satisfied, and for the long exact sequence also is (ii). So consider now a right exact sequence

$$X^0 \xrightarrow{f^0} \cdots X^{n-1} \xrightarrow{f^{n-1}} X^n \to 0$$

with X^i acyclic for $i < n$. Then we have at each level an exact sequence

$$0 \to ker(f^{k-1}) \to X^k \to ker(f^k)$$

and since X^k is F-acyclic, $R^iF(ker(f^k)) = R^{i+1}F(ker(f^{k-1}))$, so in particular

$$R^iF(X^n) = R^{i+n}F(ker(f^0))$$

Hence if $n > cd(F)$, X^n is F-acyclic, so also (iii) is satisfied, hence for lemma C.3.7, $EX1$ holds, and by the same argument as before $EX2$ holds, so RF exists, and we conclude by proposition C.4.6 with $K(\mathcal{A})$, $K^+(\mathcal{A})$ and L'. Notice that $EX1$ holds since $L \subseteq L' \cap K^+(\mathcal{A})$ □

Proposition C.4.10. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, $K^*(\mathcal{A})$ and $K^\dagger(\mathcal{B})$ be localizing subcategories and let*

$$F : K^*(A) \to K(\mathcal{B})$$
$$G : K^\dagger(B) \to K(\mathcal{C})$$

be ∂-functor.

a) Assume $F(K^*(\mathcal{A})) \subseteq K^\dagger(\mathcal{B})$, assume R^*F, $R^\dagger G$ and $R^(GF)$ exist, assume $R^*F(D^*\mathcal{A}) \subseteq D^\dagger(\mathcal{B})$. Then there exists a unique natural transformation

$$\zeta : R^*(GF) \to R^\dagger G R^* F$$

[2] i.e. there is an integer n such that $R^iF(Y) = 0$ for all $Y \in \mathcal{A} \hookrightarrow K^+(\mathcal{A})$ and all $i > n$

such that the following diagram commutes:

$$\begin{tikzcd} QGF \arrow[r, "\xi_G F"] \arrow[d, "\xi_{GF}"] & R^\dagger G Q^\dagger F \arrow[d, "R\dagger G\xi_F"] \\ R^* GFQ^* \arrow[r, "\zeta_Q"] & R^\dagger G R^* F Q^* \end{tikzcd}$$

b) *Assume $F(K^*(\mathcal{A})) \subseteq K^\dagger(\mathcal{B})$, assume that there are triangulated subcategories $L \subseteq K^*(\mathcal{A})$ and $M \subseteq K^\dagger(\mathcal{B})$ satisfying EX1 and EX2 respectively for F and G, and assume $F(L) \subseteq M$, so L satisfies 1 and 2 for GF. Hence a) holds and ζ is an isomorphism*

Proof. Straight from the definition:

a) ζ comes applying multiple times the universal property of $R^*(GF)$ to

$$\begin{tikzcd} QGF \arrow[rr, "R^\dagger G\xi_F\xi_G F"] \arrow[dr, "\xi_{GF}"'] & & R^\dagger G R^* F Q \\ & R^(GF)Q \arrow[ur, dotted, "\exists!\ \zeta_Q"'] & \end{tikzcd}$$

b) If $I \in L$, then $FI \in M$, so $\xi_G F(I)$, $\xi_F(I)$ and $\xi_G(F(I))$ are isomorphisms and every object $X^\bullet$ is quasi-isomorphic to $I^\bullet \in L$, so we can suppose $X^\bullet = Q(I^\bullet)$, hence

$$\zeta_{X^\bullet} = \zeta_{Q^*(I^\bullet)} = R^\dagger G\xi_F(I^\bullet)\xi_G(I^\bullet)\xi_{GF}(I^\bullet)^{-1}$$

So it is an isomorphism

□

Corollary C.4.11. *1. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let*

$$F : K^+(\mathcal{A}) \to K(\mathcal{B})$$
$$G : K^+(\mathcal{B}) \to K(\mathcal{C})$$

be δ-functors such that $F(K^+(\mathcal{A})) \subseteq K^+(\mathcal{B})$. Then $R^+FG \cong R^+FR^+G$

2. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, let

$$F : \mathcal{A} \to \mathcal{B}$$
$$G : \mathcal{B} \to \mathcal{C}$$

be additive functors. Assume that there exist $P_\mathcal{A} \subseteq \mathcal{A}$ and $P_\mathcal{B} \subseteq \mathcal{B}$ with properties (i), (ii) and (iv).

- *If $F(P_\mathcal{A}) \subseteq P_\mathcal{B}$. Then $R^+GF \cong R^+GR^+F$.*
- *If F, G and GF have finite cohomological dimension and F sends $P_\mathcal{A}$ into G-acyclic then $RGF = RGRF$*

Proof. Adapt arguments form corollary C.4.7 and corollary C.4.8

Remark C.4.12. Everything we did in this section can be applied to left derived functors:

Definition C.4.13. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $K^*(\mathcal{A})$ a localizing subcategory of $K(\mathcal{A})$, and let

$$F : K^*(\mathcal{A}) \to K(\mathcal{B})$$

be a ∂-functor. Let $Q^* : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ and $Q : K(\mathcal{B}) \to D(\mathcal{B})$ be the localization functors. Then the *left derived functor* of F is a ∂-functor

$$\mathcal{L}^*F : D^*(\mathcal{A}) \to D(\mathcal{B})$$

together with a natural transformation of functors from $K^*(\mathcal{A})$ to $D(\mathcal{B})$:

$$\xi : \mathcal{L}^*FQ^* \dot{\to} QF$$

with the universal property that for every δ-functor

$$G : D^*(\mathcal{A}) \to D(\mathcal{B})$$

and every natural transformation

$$\zeta : GQ^* \dot{\to} QF$$

there is a unique natural transformation $\eta : G \to \mathcal{L}^*F$ such that the following diagram commutes

$$\begin{tikzcd}
GQ^* \arrow[rr, "\zeta"] \arrow[dr, "\eta Q^*"'] & & QF \\
& \mathcal{L}^*FQ^* \arrow[ur, "\xi"'] &
\end{tikzcd}$$

By the usual argument, if $\mathcal{L}^*F$ exists it is unique up to natural isomorphism.

Then theorem C.4.5 can be restated as

Theorem C.4.14. *Let $\mathcal{A}$, $\mathcal{B}$, $K^*(\mathcal{A})$ and F as before, suppose that there is a triangulated subcategory $L \subseteq K^(\mathcal{A})$ such that*

EX1 Every object of $K^*(\mathcal{A})$ admits a quasi isomorphism from an object of L

EX2 If $P^\bullet \in L$ is acyclic, then $FP^\bullet$ is acyclic

Then F admits a right derived functor $(\mathcal{L}F, \xi)$ and for every object $P^\bullet \in L$,

$$\xi_{P^\bullet} RFQ^*(P^\bullet) \to QF(I^\bullet)$$

is an isomorphism in $D(\mathcal{B})$.

And

Corollary C.4.15. *a. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives, $F : K^-(\mathcal{A}) \to K(\mathcal{B})$ a ∂-functor, then $\mathcal{L}^-F$ exists*

b. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ an additive functor, let $P \subseteq \mathcal{A}$ such that*

(i') *Every object of $\mathcal{A}$ admits a surjection from an object of P ($\Rightarrow$ every $X^\bullet \in K^-(\mathcal{A})$) has a quasi-isomorphism from a bounded above complex $P^\bullet$ of objects of P*

(ii') *If $0 \to X \to Y \to Z \to 0$ is a short exact sequence with $Z \in P$, then $Y \in P \Leftrightarrow X \in P$*

(iv') *F preserves exact sequences of objects of P.*

Then $\mathcal{L}^-(F)$ exists

c. *If F has finite homological dimension[3], then $\mathcal{L}F$ exists and its restriction to $D^-(\mathcal{A})$ is equal to $\mathcal{L}^-F$.*

and the same for the composition

C.5 Ext, $\mathcal{R}$Hom and cup products

Let $\mathcal{A}$ be an abelian category, X and Y in $D(\mathcal{A})$. We will now study

$$\mathrm{Ext}^i(X,Y) := \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D(\mathcal{A})}(X[-i], Y)$$

Remark C.5.1. If $K^*(\mathcal{A})$ is a localizing subcategory, $X, Y \in D^*(\mathcal{A})$, then $\mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D^*(\mathcal{A})}(X, Y[i])$ since $D^*(\mathcal{A})$ is fully faithful

Proposition C.5.2. *Let $0 \to X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to 0$ an exact sequence of complexes. Then for every $V^\bullet$ we have long exact sequences*

$$\cdots \to Hom(Z^\bullet, V^\bullet[i]) \to Hom(Y^\bullet, V^\bullet[i]) \to Hom(X^\bullet, V^\bullet[i]) \to Hom(Z^\bullet, V^\bullet)[i+1] \to \cdots$$
$$\cdots \to Hom(V^\bullet, X^\bullet[i]) \to Hom(V^\bullet, Y^\bullet[i]) \to Hom(V^\bullet, Z^\bullet[i]) \to Hom(V^\bullet, X^\bullet[i+1]) \to \cdots$$

Proof. Since $Z^\bullet$ is quasi-isomprphic to $Cone(f)$, we conclude since $\mathrm{Hom}_{D(\mathcal{A})}(\_, V^\bullet)$ and $\mathrm{Hom}_{D(\mathcal{A})}(V^\bullet, \_)$ are cohomological functors. The first comes form the fact that in $D(\mathcal{A})^{op}$ the translation functor is $[-1]$ and $\mathrm{Hom}(X^\bullet[-i], V^\bullet) = Hom(X^\bullet, V^\bullet[i])$

Definition C.5.3. If $X^\bullet$ and $Y^\bullet$ are complexes of objects in $\mathcal{A}$, we define a complex

$$\mathrm{Hom}^n(X^\bullet, Y^\bullet) = \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^m, Y^{m+n})$$

and $d^n(\prod(f^m)) = \prod(f^{m+1}d_X^m + (-1)^{n+1}d_Y^{m+n}f^{m+n}$. Notice that by definition

$$d^n(\{f^m : X^m \to Y^{m+n}\}) = 0 \Leftrightarrow f^{m+1}d_X^m = (-1)^n d_Y^{m+n}f^{m+n} \Leftrightarrow f^{m+1}d_X^m = d_{Y[n]}^m f^m$$

Hence if f is a morphism of complexes , and y the same mean we can see that $f = d^{n-1}g$ iff f is null-homotopic. Hence

$$H^n(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet)$$

So we have a bi-∂-functor

$$\mathrm{Hom}^\bullet : K(\mathcal{A})^{op} \times K(\mathcal{A}) \to K(Ab)$$

[3]i.e. there is n such that for all $i < -n$ and $Y \in \mathcal{A}$ $\mathcal{L}^i F(Y) = 0$

Lemma C.5.4. *Let $X \in K\mathcal{A}$, and let $I \in K^+(\mathcal{A})$ be a complex of injective objects. Assume that either X or I is acyclic. Then $Hom_{D(\mathcal{A})}(X, Y)$ is acyclic*

Proof. Since $I[n]$ satisfies the hypothesis of the lemma for all n, it is enough to prove that any morphism $X \to I$ is null-homotopic. If X is acyclic, it comes from lemma C.3.6, if I is acyclic, it splits, hence the homotopy is the one given by the splitting. □

We will use this lemma to derive $\mathrm{Hom}^\bullet$. Let $\mathcal{A}$ be a category with enough injectives, take $L \subseteq K^+(\mathcal{A})$ be the triangulated subcategory of complexes of injective objects. Then

$$\mathrm{Hom}^\bullet(X^\bullet, \_) : K^+(\mathcal{A}) \to K(\mathcal{A})$$

sends injectives into acyclic objects, then theorem C.4.5 holds, hence we have a right derived functor, which by universal property is natural in $X^\bullet$, hence we can define a bi-∂-functor

$$R_{II}\mathrm{Hom}^\bullet : K\mathcal{A}^{op} \times D^+(\mathcal{A}) \to D(Ab)$$

Now fix $Y \in D^+(\mathcal{A})$, then Y is quasi-isomorphic to a complex I of injective objects, so $R_{II}\mathrm{Hom}^\bullet(\_, Y^\bullet) = \mathrm{Hom}^\bullet(\_, I^\bullet)$, which for the lemma preserves acyclics, so it extends to a functor

$$R_I R_{II}\mathrm{Hom}^\bullet : D(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D(Ab)$$

If $\mathcal{A}$ has enough projectives, we can construct by the same way

$$R_{II}R_I\mathrm{Hom}^\bullet : D^-(\mathcal{A})^{op} \times D(\mathcal{A}) \to D(Ab)$$

Then we notice that by definition of derived functors, we have the following lemma

Lemma C.5.5. *If*

$$T : K * (\mathcal{A}) \times K^\dagger(\mathcal{B}) \to K(\mathcal{C})$$

is a bi-∂-functor, suppose $R_I R_{II} T$ and $R_{II} R_I T$ both exist, then there is a unique natural isomorphism compatible with $\xi_{I,II}$ and $\xi_{II,I}$

Hence we will denote without ambiguity $R^*\mathrm{Hom}^\bullet$.

Theorem C.5.6 (Yoneda). *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for any $X \in D(\mathcal{A})$, $Y \in D^+(\mathcal{A})$*

$$H^i(R^+Hom^\bullet(X, Y)) = Hom_{D(\mathcal{A})}(X, Y[i])$$

Proof. Consider $s : Y \to I$ be a quasi iso to a complex of injective objects. Then by lemma C.3.6

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D(\mathcal{A})}(X, I[i]) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y[i])$$

And as we have seen

$$\mathrm{Hom}_{K(\mathcal{A})}(X, Y[i]) = H^i(\mathrm{Hom}^\bullet(X, I)) = H^i(R\mathrm{Hom}^\bullet(X, Y))$$

□

Remark C.5.7. If $\mathcal{A}$ has enough injectives, X, Y in $\mathcal{A}$, then if $Y \to I^\bullet$ is an injective resolution

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = H^i(\mathrm{Hom}(X, I^\bullet)) = H^i(R\mathrm{Hom}^\bullet(x,Y)) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i])$$

So in this case $\mathrm{Ext}^i(X,Y) = \mathrm{Ext}^i_{\mathcal{A}}(X,Y)$

Definition C.5.8. We can define a pairing

$$\mathrm{Ext}^i(X,Y) \times \mathrm{Ext}^j(Y,Z) \to \mathrm{Ext}^{i+j}(X,Z)$$

by taking the composition:

$$\begin{aligned} f &\in \mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) \\ g &\in \mathrm{Ext}^j(Y,Z) = \mathrm{Hom}_{D(\mathcal{A})}(Y, Z[j]) \Rightarrow g[i] \in \mathrm{Hom}_{D(\mathcal{A})}(Y[i], Z[i+j]) \\ f \cup g &:= g[i](f) \in \mathrm{Hom}_{D(\mathcal{A})}(X, Z[i+j]) \end{aligned}$$

If $F : D(\mathcal{A}) \to D(\mathcal{B})$ is a ∂-functor, we can also define a cup product

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \times \mathrm{Ext}^j_{\mathcal{B}}(FY, FZ) \to \mathrm{Ext}^{i+j}_{\mathcal{B}}(FX, FZ)$$

by taking the composition:

$$\begin{aligned} f \in \mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) &\Rightarrow Rf \in \mathrm{Hom}_{D(\mathcal{A})}(RFX, RFY[i]) \\ g \in \mathrm{Ext}^j(FY, FZ) &= \mathrm{Hom}_{D(\mathcal{A})}(RFY, RFZ[j]) \\ f \cup g := g[i]RF(f) \in \mathrm{Hom}_{D(\mathcal{A})}(RFX, RZ[i+j]) &= \mathrm{Ext}^i(RFX, RZ[i+j) \end{aligned}$$

C.6 Way-out functors

Definition C.6.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $F : D^*(\mathcal{A}) \to D(\mathcal{B})$ a ∂-functor. F is *way-out right* if for all $n_1 \in \mathbb{Z}$ there exists $n_2 \in \mathbb{Z}$ such that for all $X \in D(\mathcal{A})$ such that $H^i(X) = 0$ for $i < n_2$, then $R^i(X) = 0$ for $i < n_1$. Similarly we define *way-out left* and *way-out in both directions*

Example C.6.2. Let $F : \mathcal{A} \to \mathcal{B}$ a situation as in corollary C.4.7 or corollary C.4.8, then R^+F is way-out right. If F is as in corollary C.4.9, then RF is way-out in both directions.

Definition C.6.3. We define two truncated complexes:

$$\begin{aligned} \tau_{>n}(X^\bullet) &= \cdots 0 \to X^{n+1} \to \cdots \\ \tau_{\leq n}(X^\bullet) &= \cdots \to X^n \to 0 \cdots \\ \sigma_{>n}(X^\bullet) &= \cdots 0 \to Im(d^n) \to X^{n+1} \cdots \\ \sigma_{\leq n}(X^\bullet) &= \cdots X^{n-1} \to Ker(d^n) \to 0 \cdots \end{aligned}$$

Notice that $H^i(\sigma_{>n}(X)) = H^i(X)$ if $i > n$ and $H^i(\sigma \geq nX) = H^i(X)$, and we have triangles in $D(A)$:

$$(\tau_{>n}(X), \tau_{>n-1}(X), X^n) \qquad\qquad (\sigma_{>n-1}(X), \sigma_{>n}(X), H^n(X))$$

given by the exact sequences

$$0 \to \tau_{>n}(X) \to \tau_{\geq n}(X) \to X^n \to 0$$
$$0 \to H^n(X) \to \sigma'_{\geq 1}(X) \to \sigma_{>n}(X) \to 0$$

With $\sigma'_{\geq n}(X) = 0 \to X^n/im(d^{n-1} \to X^{n+1} \to \cdots$ is quasi isomorphic to $\sigma_{>n-1}$[4]

Definition C.6.4. A subcategory is called thick if it is closed by extensions. In particular, if $\mathcal{A}'$ is a thick abelian subcategory of an abelian category $\mathcal{A}$, then it is closed for short exact sequence (i.e. every time two terms of a short exact sequence are in $\mathcal{A}'$, also the third one is in $\mathcal{A}'$).

Then one can define subcategory $K_{\mathcal{A}'}(\mathcal{A})$ as the full subcategory of $K(\mathcal{A})$ whose objects are complexes $X^\bullet$ such that $H^i(X) \in \mathcal{A}'$. The thickness makes it a full triangulated subcategory (i.e. every time two edges of a triangle are in $K_{\mathcal{A}'}(\mathcal{A})$, so is the third one). We also can define $D_{\mathcal{A}'}(\mathcal{A}) = K_{\mathcal{A}'}(\mathcal{A})_{Qis}$ and by proposition C.2.6 it is the full subcategory of $D(\mathcal{A})$ whose objects are complexes $X^\bullet$ such that $H^i(X) \in \mathcal{A}'$. Similarly one can define $K^+_{\mathcal{A}'}(\mathcal{A})$, $D^+_{\mathcal{A}'}(\mathcal{A})$ et cetera.

Lemma C.6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$ and let F, G be ∂-functors $D^+_{\mathcal{A}'}(\mathcal{A}) \to D^(\mathcal{B})$, and let $\eta : F \dot{\to} G$ be a natural transformation. Then*

(i) Assume that $\eta(X)$ is an isomorphism for all $X \in \mathcal{A}'$, then $\eta(X^\bullet)$ is an isomoprhism for all $X^\bullet \in D^b_{\mathcal{A}'}(\mathcal{A})$.

(ii) Assume that $\eta(X)$ is an isomorphism for all $X \in \mathcal{A}'$ and that F and G are way-out right. Then $\eta X^\bullet$ is an isomoprhism for all $X^\bullet \in D^+_{\mathcal{A}'}(\mathcal{A})$

(iii) Assume that $\eta(X)$ is an isomorphism for all $X \in \mathcal{A}'$ and that F and G are way-out in both directions. Then $\eta X^\bullet$ is an isomoprhism for all $X^\bullet \in D_{\mathcal{A}'}(\mathcal{A})$

(iv) Let $P \subseteq \mathcal{A}'$ such that every object of $\mathcal{A}'$ embeds into an object of P. Assume $\eta(X)$ is an isomorphism for every $X \in P$ and that F and G are way-out right. Then $\eta(X)$ is an isomorphism for all $X \in \mathcal{A}'$

Proof. (i) Let $X \in D^b_{\mathcal{A}'}(\mathcal{A})$, then, $X \to \sigma_{>n}(X)$ is a quasi isomorphism for $n << 0$, so it is enough to prove by descending induction that

$$\eta(\sigma_{>n}(X^\bullet)) : F(\sigma_{>n}(X^\bullet)) \to G(\sigma_{>n}(X^\bullet))$$

is a quasi isomorphism. Since $X^\bullet$ has bounded cohomology, for $n >> 0$ $\sigma_{>n}(X)$ is exact, hence $\eta(\sigma_{>n}(X^\bullet)) = 0$ is a quasi isomorphism. The induction step follows from the fact that in the morphism of triangles

$$(\eta(\sigma_{\geq n}(X^\bullet)), \eta(\sigma_{>n}(X^\bullet)), \eta(H^n(X))) : (F\sigma_{\geq}(X), F\sigma_{>n}(X), FH^n(X)) \to (G\sigma_{\geq}(X), G\sigma_{>n}(X), GH^n(X))$$

$\eta(H^n(X))$ is an iso by hypothesis and $\eta(\sigma_{>n}(X^\bullet)$ is an iso by induction hypothesis, so is $\eta(\sigma_{\geq n}(X^\bullet)$

[4] $d^n(im(d^{n-1}) = 0$, so $im(d^n) = im(X^n/im(d^{n-1})$.

(ii) It is enough to show that

$$H^j(\eta(X^\bullet)) : H^j(FX^\bullet) \to H^j(GX^\bullet)$$

is an isomorphism for all j. Let $n_1 > j+2$ and $n_2 = \min(n_2^F, n_2^G)$ as in the definition of way-out functors. Consider the triangle $(\sigma_{>n_2}(X^\bullet), X^\bullet, \sigma_{\leq n_2}(X^\bullet))$ Since $H^i(\sigma_{>n_2}(X)) = 0$ for $i \leq n_2$, by the way out property

$$H^i(F\sigma_{>n_2}(X)) = H^i(G\sigma_{>n_2}(X)) = 0 \qquad\qquad i < n_n$$

In particular they are zero for $i = j, j+1$, hence we have isomorphisms

$$H^j(F\sigma_{\leq n_2}(X)) \cong H^j(FX) \qquad\qquad H^j(G\sigma_{\leq n_2}(X)) \cong H^j(GX)$$

And since $\sigma_{\leq n_2}(X) \in D^b_{\mathcal{A}'}(\mathcal{A})$, for the previous point we have that $\eta(\sigma_{\leq n_2}(X))$ is a quasi isomorphism, hence $H^j(\eta(X^\bullet))$ is an isomorphism.

(iii) Apply the previous idea to $\sigma_{>0}(X^\bullet)$ and $\sigma_{\leq 0}(X^\bullet)$, and glue together with the exact sequence.

(iv) Consider $X \to I^\bullet$ a resolution by objects in P, it is enough to show that $\eta(I^\bullet)$ is a quasi isomorphism for all complexes in P since $\eta(X) = H^0(\eta(I^\bullet))$, and the same technique as in (ii) shows that it is sufficient to show it for $I^\bullet$ bounded, and if we proceed as in (i) but considering the triangle $(\tau_{>n}, \tau_{\geq n}, X^n)$ we have the induction step.

□

With the same techniques we can prove that:

Proposition C.6.6. *If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}'$ and $\mathcal{B}'$ are thick abelian subcategories, then if $F : D_{\mathcal{A}'}(\mathcal{A}) \to D(\mathcal{B})$. Then*

(i) Assume $FX \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$, then $FX^\bullet \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in D^b_{\mathcal{A}'}(\mathcal{A})$

(ii) Assume $FX \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$ and F is way-out right, then $FX^\bullet \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in D^+_{\mathcal{A}'}(\mathcal{A})$

(iii) If F is way-out in both direction, then $FX^\bullet \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in D_{\mathcal{A}'}(\mathcal{A})$

(iv) Let $P \subseteq \mathcal{A}'$ such that every object of $\mathcal{A}'$ embeds into an object of P. Assume $FX \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in P$ and F is way-out right, then $FX \in D_{\mathcal{B}'}(\mathcal{B})$ for all $X \in \mathcal{A}'$

Proposition C.6.7. *Let $\mathcal{A}$ be an abelian category with enough injectives, let $X^\bullet \in K^+(\mathcal{A})$. Then the following are equivalent*

(i) $X^\bullet$ admits a quasi isomorphism into a bounded complex of injective objects

(ii) $RHom^\bullet(\_, X^\bullet)$ is way-out in both directions

(iii) There exists n_0 such that $Ext^i(Y, X^\bullet) = 0$ for all $Y \in \mathcal{A}$ and $i > n_0$

Proof. $(i) \Rightarrow (ii)$ We have that

$$RHom(\_, X^\bullet) = Hom(\_, I^\bullet)$$

and since $I^\bullet$ is bounded it is way-out in both directions.

$(ii) \Rightarrow (iii)$ Consider in the definition of way-out left $n_1 = 0$. Then there exists n_0 such that for all $Y^\bullet$ such that $H^i(Y^\bullet) = 0$ for $i < n_0$, $H^i(Hom^\bullet(Y^\bullet, X^\bullet)) = 0$ for $i > 0$. Then take the complex $Y[-n_0]$, by definition $H^i(Y[-n_0]) = 0$ for $i < n_0$, hence for all $i > 0$

$$0 = H^i(Hom^\bullet(Y^\bullet[-n_0], X^\bullet)) = H^{i+n_0}(Hom^\bullet(Y, X^\bullet)) = Ext^{i+n_0}(Y, X^\bullet)$$

$(iii) \Rightarrow (i)$ Consider $X^\bullet \to I^\bullet$ a quasi isomorphism to a complex of injective objects bounded below.

Claim $H^i(I^\bullet) = 0$ for $i > n_0$: suppose that there exists $m > n_0$ such that $H^i(I^\bullet) \neq 0$. Hence there exists a $Y \in \mathcal{A}$ such that

$$Hom(Y, B^m(I^\bullet)) \to Hom(Y, Z^m(I^\bullet))$$

is a mono non iso. On the other hand,

$$Z^m Hom^\bullet(Y, (I^\bullet)) = Hom_{Ch(\mathcal{A})}(Y, I^\bullet[m]) = \{f : Y \to I^m : d^m f = 0\} = Hom_{\mathcal{A}}(Y, Z^m(I^\bullet))$$

and

$$B^m Hom^\bullet(Y, (I^\bullet)) = \{f : \exists\ s : Y \to (I^\bullet)^{m-1} : sd^{m-1} = f\} \to Hom(Y, B^m(I^\bullet))$$

and we have a diagram

$$\begin{tikzcd} B^m(Hom^\bullet(Y, (I^\bullet))) \arrow[r] \arrow[d] & Z^m B^m(Hom^\bullet(Y, (I^\bullet))) \arrow[d, "\sim"] \\ Hom(Y, B^m(I^\bullet)) \arrow[r] & Hom(Y, Z^m(I^\bullet)) \end{tikzcd}$$

But since $H^m(Hom^\bullet(Y, (I^\bullet))) = Ext^m(Y, X^\bullet) = 0$ the top arrow is an isomorphism, which implies that the bottom arrow is epi, which is a contradiction.

Hence $H^i(I^\bullet) = 0$, so for $n > n_0$

$$\sigma_{\leq n} I^\bullet \to I^\bullet$$

is a quasi isomorphism. To conclude, we need to show that $\sigma_{\leq n} I^\bullet$ is a complex of injective objects, in particular we need to show that $Z^{n+1}(I^\bullet) = B^{n+1}(I^\bullet)$ is injective for $n > n_0$.
Consider the exact sequence

$$0 \to \sigma_{\leq n}(I^\bullet) \to \tau > n(I^\bullet) \to B^{n+1}(I^\bullet)[-n] \to 0$$

which gives for all Y a long exact sequence

$$\mathrm{Ext}^{n+1}(Y, \tau_{\leq n}(I^\bullet)) \to \mathrm{Ext}^{n+1}(Y, B^{n+1}(I^\bullet)[-n]) \to \mathrm{Ext}^{n+2}(Y, \sigma_{\leq n}(I^\bullet))$$

Since $\tau_{\leq n}(I^\bullet)[n+1]_0 = 0$ and $\tau_{\leq n}(X^\bullet)$ is a complex of injective objects,

$$\mathrm{Ext}^{n+1}(Y, \tau_{\leq n}(X)) = \mathrm{Hom}_{K(\mathcal{A})}(Y, \tau_{\leq n}(X^\bullet)[n+1]) = 0$$

And since $\sigma_{\leq n}(I^\bullet)$ is quasi isomorphic to $X^\bullet$, by hypothesis $\mathrm{Ext}^{n+2}(Y, \sigma_{\leq n}(I^\bullet)) = 0$, so

$$\mathrm{Ext}^1(Y, B^{n+1}(I^\bullet)) = \mathrm{Ext}^{n+1}(Y, B^{n+1}(I^\bullet)[-n]) = 0$$

Then $B^{n+1}(I^\bullet)$ is injective.

□

An object X that satisfies the three equivalent conditions of proposition C.6.7 is said to have *finite injective dimension*.

C.7 Application to schemes

C.7.1 The derived tensor product

Let X be a site, Λ a ring, $D(X, \Lambda)$ the derived category of $Sh(X, \Lambda)$. With the long exact sequence one can see that the full subcategory $Fl(X, \Lambda)$ of flat sheaves is a triangulated subcategory of $Sh(X, \Lambda)$ satisfying $EX1$ and $EX2$. If $F, G \in D(X, \Lambda)$, one can consider the double complex $K^{pq} = F^p \otimes_\Lambda G^q$, and define $F^\bullet \otimes_\Lambda G^\bullet$ as the total complex associate, so there is a bifunctor

$$\otimes : K(X, \Lambda) \otimes K(X, \Lambda) \to K(X, \Lambda)$$

Lemma C.7.1. *If $F^\bullet \in K(X, \Lambda)$ and $G^\bullet \in Fl(X, \Lambda)$ such that*

1. *$F^\bullet$ is acyclic OR $G^\bullet$ is acyclic*
2. *$F^\bullet$ is bounded above OR $G^\bullet$ is bounded.*

Then $F^\bullet \otimes_\Lambda G^\bullet$ is acyclic

Proof. By considering the two hypercohomology spectral sequences

$$'E_2^{pq} = H_I^p H_{II}^q(K) \Rightarrow \mathbb{H}^{p+q}(K)''E_2^{pq} = H_{II}^p H_I^q(K) \Rightarrow \mathbb{H}^{p+q}(K)$$

The hypothesis 2. says that it is enough to prove that for $(p,q) \neq (0,0)$ either $'E_2^{pq} = 0$ or $''E_2^{pq} = 0$. If G is acyclic, then $B^q(G) = Z^q(G)$ is flat for each q, so

$$\mathrm{Tor}_1(F^p \otimes B^q) = 0 \to F^p \otimes Z^q \to F^p \otimes_\Lambda G^q \to F^p \otimes B^q \to 0$$

is exact for all q, so $'E_2^{pq} = 0$ for $(p,q) \neq (0,0)$.
On the other hand, since G^q is flat $F^\bullet \otimes_\Lambda G^q$ is acyclic, so $''E_2^{pq} = 0$ for $(p,q) \neq (0,0)$.

so one can consider

$$\_\otimes^L_\Lambda\_ : D^-(X,\Lambda)\times D^-(X,\Lambda)\to D^-(X,\Lambda)$$

the derived functor $\mathcal{L}_{II}\mathcal{L}_I(\otimes)=\mathcal{L}_{II}\mathcal{L}_I(\otimes)$, and the hypertor

$$\mathbb{T}\mathrm{or}_i(F^\bullet,G^\bullet)=H^{-i}(F^\bullet,G^\bullet)$$

Remark C.7.2. If everything is well defined, there is an adjunction $\_\otimes^L_\Lambda K\dashv R\mathrm{Hom}(K,\_)$, in fact considering $X\to I$ and $P\to L$ quasi-isomorphic complexes respectively of injective and flat sheaves, then

$$\begin{aligned}\mathrm{Hom}_{D(X,\Lambda)}(K\otimes^L_\Lambda L,X)&\cong\mathrm{Hom}_{D(X,\Lambda)}(K\otimes_\Lambda P,I)\cong\mathrm{Hom}_{D(X,\Lambda)}(P,\mathrm{Hom}(K,I))\\&\cong\mathrm{Hom}_{D(X,\Lambda)}(L,R\mathrm{Hom}(K,X))\end{aligned}$$

There is an analogue of proposition C.6.7 for $\mathbb{T}\mathrm{or}$:

Proposition C.7.3. *Let $F^\bullet\in D^b(X,\Lambda)$, then TFAE*

(i) There is a quasi isomorphism $F'^\bullet\to F^\bullet$ such that $F'^\bullet\in D^b_{Fl}(X,\Lambda)$

(ii) The functor $F^\bullet\otimes^{\mathbb{L}}_\Lambda\_$ is way out in both directions

(iii) There exists n such that $\mathbb{T}\mathrm{or}_i(F^\bullet,G)=0$ for all $i>n$ and $G\in Sh(X,\Lambda)$

Proof. The proof is exactly the same as in proposition C.6.7, except from the fact that in $(iii)\to(i)$ the flat resolution $P^\bullet\to F^\bullet$ given by lemma C.3.7 is not bounded below, but one can consider the commutative diagram

$$\begin{tikzcd} P^\bullet \arrow[r] \arrow[d] & F^\bullet \arrow[d] \\ \sigma_{>n}(P^\bullet) \arrow[r] & F^\bullet \end{tikzcd}$$

and since $F^\bullet$ is bounded below, $\sigma_{>n}(F^\bullet)=X^\bullet$, hence the proof follows considering the bounded below quasi-isomorphic complex of flat modules $\sigma_{>n}(P^\bullet)$[5].

Definition C.7.4. An object F that satisfies the three equivalent conditions of proposition C.7.3 is said to have *finite Tor dimension.*

C.7.2 The Projection Formula

Let now $f:X\to Y$ and $g:Y\to Z$ morphism of schemes, then since f_* preserves injectives

$$R^+g_*R^+f_*F\cong R^+(gf)_*F$$

and if f_* has finite cohomological dimension

$$Rg_*Rf_*F\cong R(gf)_*F$$

[5] $Im(d^n)$ is flat for condition (iii) of lemma C.3.7, since for $n<<0$ $X^n=0$, so $\sigma_{<n}(P^\bullet)$ is acyclic

So consider $F \in Sh(X, \Lambda)$ $G \in Sh(Y, \Lambda)$ and $f : X \to Y$ a morphism of schemes. Then there is a canonical map

$$G \otimes_\Lambda f * F \to f_* f^* G \otimes_\Lambda f_* F = f_*(f^* G \otimes_\Lambda F)$$

Remark C.7.5. f^* preserves flat modules: in fact if G is flat and $F \rightarrowtail F'$ is a mono, then for every $\bar{x}$ is a geometric point

$$(f^* G \otimes_\Lambda F)_{\bar{x}} \cong G_{f\bar{x}} \otimes_\Lambda F_{\bar{x}} \xrightarrow{f_x \otimes 1} G_{f\bar{x}} \otimes_\Lambda F'_{\bar{x}} = (G \otimes_\Lambda F')_{\bar{x}}$$

is mono. Hence, since f^* is exact, we have that if $P^\bullet \to F^\bullet$ is a quasi isomorphic flat complex, then

$$(f^* F^\bullet \otimes^L_\Lambda G^\bullet) = (f^* P^\bullet \otimes_\Lambda G^\bullet)$$

Lemma C.7.6. *Let $F^\bullet \in D(X, \Lambda)$, $G^\bullet \in D(Y, \Lambda)$ and $f : X \to Y$. If one of these condition is satisfied*

a. f_* has finite cohomological dimension, $F^\bullet \in D^-(X, \Lambda)$, $G^\bullet \in D^-(Y, \Lambda)$

b. $G^\bullet \in D^b(Y, \Lambda)$ has finite Tor dimension and $F^\bullet \in D^+(X, \Lambda)$.

c. f_* has finite cohomological dimension, $G^\bullet \in D^b(Y, \Lambda)$ has finite Tor dimension

Then there is a canonical morphism

$$G^\bullet \otimes^L_\Lambda Rf_* F^\bullet \to Rf_*(f^* G^\bullet \otimes^L_\Lambda F^\bullet)$$

Proof. The idea is to take quasi isomorphisms $F^\bullet \to I^\bullet$, $P^\bullet \to G^\bullet$ and $f^* P^\bullet \otimes_\Lambda I^\bullet \to J^\bullet$ such that the derived functors are well defined in order to have the morphisms:

$$\begin{aligned} G^\bullet \otimes^L_\Lambda Rf_* F^\bullet &\cong P^\bullet \otimes_\Lambda f_* I^\bullet \\ &\to f_*(f^* P^\bullet \otimes_\Lambda I^\bullet) \\ &\to f_* J^\bullet \cong Rf_*(f^* P^\bullet \otimes_\Lambda I^\bullet) \\ &\cong Rf_*(f^* F^\bullet \otimes^L_\Lambda G^\bullet) \end{aligned}$$

a. $I^\bullet$ and $J^\bullet$ are complexes of f_*-acyclic sheaves and $P^\bullet$ is a bounded above comlplex of flat sheaves. Then the derived tensor product is well defined on $D^-(X, \Lambda)$ and Rf_* is well defined on $D(X, \Lambda)$.

b. Since $F^\bullet$ and $G^\bullet$ are bounded below, $G^\bullet \otimes_\Lambda f^* F^\bullet$ is bounded below. $I^\bullet$ and $J^\bullet$ are bounded below complexes of injective sheaves and $P^\bullet$ is a bounded comlplex of flat sheaves. Then since $G^\bullet$ has finite Tor dimension the derived tensor product is well defined on $D(X, \Lambda)$ and Rf_* is well defined on $D^+(X, \Lambda)$.

c. $I^\bullet$ and $J^\bullet$ are complexes of f_*-acyclic sheaves and $P^\bullet$ is a bounded comlplex of flat sheaves. Then since $G^\bullet$ has finite Tor dimension the derived tensor product is well defined on $D(X, \Lambda)$ and Rf_* is well defined on $D(X, \Lambda)$.

□

Remark C.7.7. Consider $\mathcal{A}$ the subcategory of $Sh(X, \Lambda)$ of locally constant sheaves. It is easy to see using the long exact sequence that it is an abelian and thick subcategory. Let us denote

$$D^*_{lc}(X, \Lambda) := D^*_{\mathcal{A}}(X, \Lambda)$$

Proposition C.7.8 (Projection formula). *Let $f : X \to Y$ with f_* of finite cohomological dimension. Then for any $F^\bullet \in D^-(X, \Lambda)$ and any $G^\bullet \in D^-_{lc}(Y, \Lambda)$, there is a canonical isomorphism*

$$G^\bullet \otimes^L_\Lambda Rf_*F^\bullet \xrightarrow{\sim} Rf_*(f^*G^\bullet \otimes^L_\Lambda F^\bullet)$$

Proof. Since $\otimes$ is right exact and f_* has finite cohomological dimension, the functors $\_ \otimes^L_\Lambda Rf_*F^\bullet$ and $Rf_*(f^*G^\bullet \otimes^L_\Lambda F^\bullet)$ are way-out right. So for lemma C.6.5 it is enough to show that the theorem holds for every locally constant sheaf G, and since being isomorphic is a local property, we may assume G constant. Let M be the Λ-moudule associated.
Let $L^\bullet \to M \to 0$ a free resolution of M, and consider $F \to I^\bullet$ be a quasi isomorphism into a bounded above complex of f_*-acyclic Λ-modules, andince L^p is free, f^*L^p is locally constant locally free, hence $f^*L^p \otimes F'q$ is f_*-acyclic. Then

$$G \otimes^L_\Lambda Rf_*F \cong L^\bullet \otimes_A f_*I^\bullet$$
$$Rf_*(f^*G \otimes^L_\Lambda Rf_*F) \cong f_*(f^*L^\bullet \otimes_A I^\bullet)$$

And since $L^\bullet$ is free, $L^\bullet \to f_*f^*L^\bullet$ is an isomorphism, hence

$$L^\bullet \otimes_\Lambda f_*F \to_* f^*L^\bullet \otimes_\Lambda f_*F$$

is an isomorphism. □

C.7.3 Cohomology with support

If $j : U \hookrightarrow X$ is an open immersion and $i : Z = X \setminus U \to X$ is the closed immersion of the complementary. Recall the definition of proper support cohomology as the derived functor of

$$\Gamma_Z(X, F) = Ker(F \to j_*j^*F) = \{s \in F(X) : supp(s) \subseteq Z\}$$

If I is an injective sheaf, it is flasque, so

$$I \to j_*j^*I$$

is epi. In particular, by the mapping cylinder and the exactness of i^*, for every injective sheaf we have an exact sequence in $Sh(Z, \Lambda)$

$$0 \to i^!I \to i^*I \to i^*j_*j^*I \to 0$$

By applying the exact functor i_* and using the adjunction we have in $Sh(X, \Lambda)$ an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & i_*i^!I \arrow[r] \arrow[d, equal] & i_*i^*I \arrow[r] \arrow[d] & i_*i^*j_*j^*I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i_*i^!I \arrow[r] & I \arrow[r] & j_*j^*I \arrow[r] & 0
\end{tikzcd}$$

Hence, for every $K \in D(Z, \Lambda)$, since i^* and j^* are exact we have the triangle:

$$i_* Ri^! K \to K \to Rj_* j^* K \to$$

By applying the exact functor i_* and using the adjunction we have in $D(X, \Lambda)$ a morphism of triangles

$$\begin{tikzcd} i_* Ri^! F \arrow[r] \arrow[d, equal] & i_* i^* K \arrow[r] \arrow[d] & i_* i^* Rj_* j^* K \arrow[r] \arrow[d] & {} \\ i_* Ri^! F \arrow[r] & K \arrow[r] & i_* i^* Rj_* j^* K \arrow[r] & {} \end{tikzcd}$$

In particular, by applying $\mathrm{Hom}_{D(Z,\Lambda)}(\mathbb{Z}_Z, \_)$, since $\mathbb{Z}_Z = i^* \mathbb{Z}_X$, $\mathbb{Z}_U = j^* \mathbb{Z}_X$, i^* and i_* are exact, i_* is fully faithful so $i^* i_* \cong id$ and $j^* \vdash Rj_*$ hence we have the triangle in $D(\Lambda)$

$$\mathrm{Hom}_{D(Z,\Lambda)}(\mathbb{Z}_Z, Ri^! K) \to \mathrm{Hom}_{D(X,\Lambda)}(\mathbb{Z}_X, K) \to \mathrm{Hom}_{D(X,\Lambda)}(\mathbb{Z}_U, j^* K) \to$$

Hence we have a long exact sequence

$$H^r(Z, Ri^! K) \to H^r(X, K) \to H^r(U, j^* K)$$

In particular, by definition, $H^r_Z(X, F) = H^r(Z, Ri^! F) = \mathrm{Ext}^r_X(i_* \mathbb{Z}_Z, F)$ and we have a long exact sequence

$$H^r_Z(X, F) \to H^r(X, F) \to H^r(U, j^* F) \to$$

Proposition C.7.9 (Excision). *If $\pi : X' \to X$ is Ã©tale and $Z' \subseteq X'$ is closed such that $Z = \pi(Z')$ is closed in X, $\pi_{Z'} : Z' \to Z$ is an isomorphism and $\pi(U') \subset U$ where U' and U are the complementary open subsets. Then for any F the canonical map induces an isomorphism $H^r_Z(X, F) \cong H^r_{Z'}(X', \pi^* F)$.*

Proof. The canonical map induces a morphism of triangles

$$\begin{tikzcd} R\Gamma_Z(X, F) \arrow[r] \arrow[d] & R\Gamma(X, F) \arrow[r] \arrow[d] & R\Gamma(U, F) \arrow[r] \arrow[d] & {} \\ R\Gamma_{Z'}(X', \pi^* F) \arrow[r] & R\Gamma(X', \pi^* F) \arrow[r] & R\Gamma(U, \pi^* F) \arrow[r] & {} \end{tikzcd}$$

Since if $j : U \to X$ is the open immersion, we have that if I is injective the following diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d] & j_* j^* I \arrow[d] \\ \pi_* \pi^* I \arrow[r] & j_* j^* \pi_* \pi^* I \end{tikzcd}$$

is a pullback by the universal properties of the adjunction, hence since i^* is exact we have again an pullback, so if $i : Z \to X$, $i' : Z' \to X'$ are the closed immersions, we have an isomorphism induced to the kernels $i^! I \cong i^! \pi_* \pi^* I$, which induces a quasi isomorphism

$$\mathrm{Hom}_{D(Z)}(\mathbb{Z}, Ri^! F) \cong \mathrm{Hom}_{D(Z)}(\mathbb{Z}, Ri^! R\pi_* \pi^* F) \cong \mathrm{Hom}_{D(X')}(\pi^* i_* \mathbb{Z}, \pi^* F)$$

And since $\pi^* i_*(\mathbb{Z}_Z) = (i')_*(\mathbb{Z}_{Z'})$ since π is Ã©tale, we have

$$H^r_Z(X, F) = \mathrm{Hom}_{D(X)}(i_* \mathbb{Z}, F[r]) \cong \mathrm{Hom}_{D(X')}(i'_* \mathbb{Z}, \pi^* F[r]) = H^r_{Z'}(Z', \pi^* F)$$

□

C.8 Useful spectral sequences

C.8.1 The Ext spectral sequence for Ãl'tale cohomology

Let X be a scheme and $\pi : Y \to X$ be a Galois covering with Galois group G. Then since coverings are a Galois category for every G-module M there is a unique locally constant constructible sheaf F_M such that $F_M(Y) = M$ as G-modules.

Lemma C.8.1. *If N and P are sheaves on X and M a G-module, there is a canonical iso*

$$Hom_G(M, Hom_Y(N, P)) \cong Hom_X(M \otimes N, P)$$

Proof. Since $\mathrm{Hom}_Y(M, \mathcal{H}om(N, P)) \cong \mathrm{Hom}_Y(M \otimes N, P)$ by adjunction, and since M is constant on Y we have also the adjunction between constant and global section, so in degree zero

$$\mathrm{Hom}_Y(M, \mathcal{H}om(N, P)) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_Y(N, P))$$

By taking the G-invariants we have $\mathrm{Hom}_Y(M \otimes N, P)^G = \mathrm{Hom}_X(M \otimes N, P)$ and $\mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_Y(N, P))^G = \mathrm{Hom}_G(M, \mathrm{Hom}_Y(N, P))$ □

Lemma C.8.2. *Let I be injective and F flat, then $Hom_Y(F, I)$ is injective as G-module*

Proof. Since I is injective, $R\mathrm{Hom}_G(\_, \mathrm{Hom}(F, I)) = R\mathrm{Hom}_G(\_, R\mathrm{Hom}(F, I))$, so by previous theorem: $\mathrm{Hom}_G(\_, \mathrm{Hom}(F, I)) = \mathrm{Hom}_X(\_ \otimes F, I)$ and by hypothesis it is $R\mathrm{Hom}_X(\_ \otimes^L F, I)$, so it is exact. □

So $\mathrm{Hom}_Y(\_, \_)$ sends flats and injectives into G-acyclics, so on D^b_{fl} we can derive the composition:

$$R\mathrm{Hom}_G(M, R\mathrm{Hom}_Y(N, P)) = R\mathrm{Hom}_X(M \otimes N, P)$$

In particular, we have that if $M \otimes N = M \otimes^L N$, we have by the same idea

$$R\mathrm{Hom}_G(M, R\mathrm{Hom}_Y(N, P)) = R\mathrm{Hom}_X(M \otimes N, P)$$

i.e. a spectral sequence

$$\mathrm{Ext}^p_G(M, \mathrm{Ext}^q_Y(N, P)) \Rightarrow \mathrm{Ext}^{p+q}_X(M \otimes N, P)$$

Suppose now $M \otimes^L N = M \otimes N$, so we have

$$R\mathcal{H}om_G(M, R\mathcal{H}om_Y(N, P)) = R\mathcal{H}om_G(M, \mathcal{H}om_Y(N, I)) \cong \mathcal{H}om_X(M \otimes N, I) = R\mathcal{H}om_X(M \otimes^L N, P)$$

So we have the theorem:

Theorem C.8.3. *If M is a G-module, N and P sheaves on X, such that $M \otimes^L N = M \otimes N$, we have a spectral sequence*

$$Ext^p_G(M, Ext^q_Y(N, P)) \Rightarrow Ext^{p+q}_X(M \otimes N, P)$$

C.8.2 The Ext spectral sequence for G-modules

This spectral sequence can easily be deduced from the above calculations, but it can also be deduced without the use of Ãľtale topology. I illustrate this approach.
Throughout this section, G will be a profinite group. By a torsion-free G-module, we mean a G-module that is torsion-free as an abelian group. Let G be a topological group and let M and N be G-modules. Then consider $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ as a $\mathbb{Z}[G]$-module with action given by

$$\sigma(f) : m \mapsto \sigma f(\sigma^{-1} m)$$

In general it is not a discrete G-module. So for H a closed normal subgroup of G, we may define

$$\mathcal{H}om_H(M,N) := \bigcup_{\substack{H \le U \le G \\ \text{open}}} \mathrm{Hom}_{\mathbb{Z}}(M,N)^U$$

By definition now $\mathcal{H}om_H(M,N)$ is a discrete G/H-module, and we define $\mathcal{E}xt^r_H(M,\_)$ to be the right derived functor of $\mathcal{H}om_H(M,\_)$. If $H = 1$, I will simply write

$$\mathcal{H}om(M,N) = \bigcup_{\substack{U \le G \\ \text{open}}} \mathrm{Hom}_{\mathbb{Z}}(M,N)^U$$

If M is a finitely generated $\mathbb{Z}[G]$-module, then let $\{e_1 \dots e_n\}$ be its generators, then

$$f(a_1 e_1 + \dots + a_n e_n) = a_1 f(e_1) + \dots + a_n f(e_n)$$

So $U = \bigcap_i (Stab(e_i) \cap Stab(f(e_i)))$ is a nonempty open subgroup of G and $f \in \mathrm{Hom}_{\mathbb{Z}}(M,N)^U$. In particular $\mathcal{E}xt^r(M,N) = \mathrm{Ext}^r(M,N)$

Lemma C.8.4. *For any G-modules N and P and G/H-module M, there is a canonical isomorphism*

$$Hom_{G/H}(M, \mathcal{H}om_H(N,P)) \cong Hom_G(M \otimes_{\mathbb{Z}} N, P)$$

Proof. We have $\mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(N,P)) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} N, P)$. Taking the G invariants, on the left:

$$\mathrm{Hom}_G(M, \mathrm{Hom}_{\mathbb{Z}}(N,P)) = {}^6\mathrm{Hom}_{G/H}(M, \mathrm{Hom}_{\mathbb{Z}}(N,P)) = {}^7\mathrm{Hom}_{G/H}(M, \mathcal{H}om_{\mathbb{Z}}(N,P))$$

On the right we simply have $\mathrm{Hom}_G(M \otimes_{\mathbb{Z}} N, P)$

Lemma C.8.5. *Let N and I be G-modules with I injective, and let M be a G/H-module. Then there is a canonical isomorphism*

$$Ext^r_{G/H}(M, \mathcal{H}om_H(N,I)) \cong Hom_G(Tor^{\mathbb{Z}}_r(M,N), I)$$

[6] M is a G/H-module
[7] M is discrete

Proof. Since $\mathbb{Z}$ is a PID, $\mathrm{Tor}_r^{\mathbb{Z}}(M,N)=0$ for all $r\geq 2$ and we have that if N is torsion-free, by lemma C.8.4

$$\mathrm{Hom}_{G/H}(\_,\mathcal{H}om_H(N,I))\cong \mathrm{Hom}_G(\_\otimes_{\mathbb{Z}} N,I)$$

And we have that $\_\otimes_{\mathbb{Z}} N$, exact since N is torsion-free, hence flat, and $\mathrm{Hom}_G(\_,I)$, exact since I is injective, hence $\mathcal{H}om_H(N,I)$ is an injective G/H-module. Take $0\to N_1\to N_0\to N\to 0$ a torsion-free resolution of N, consider

$$0\to \mathrm{Tor}_1^{\mathbb{Z}}(M,N)\to M\otimes_{\mathbb{Z}} N_1\to M\otimes_{\mathbb{Z}} N_0\to M\otimes_{\mathbb{Z}} N\to 0$$

Since for all U open subgroups of G, $\mathbb{Z}[G/U]$ is a free $\mathbb{Z}$-modules, $\_\otimes_{\mathbb{Z}}\mathbb{Z}[G/U]$ is exact, and so

$$\mathrm{Hom}_G(\_\otimes_{\mathbb{Z}}\mathbb{Z}[G/U],I)=\mathrm{Hom}_U(\_,I)$$

is exact since I is injective. Hence the exact sequence

$$0\to \mathrm{Hom}_U(N,I)\to \mathrm{Hom}_U(N_0,I)\to \mathrm{Hom}_U(N_1,I)\to 0$$

is an injective resolution of $\mathrm{Hom}_G(N,I)$. Taking the limit over all U containing H, we have that

$$0\to \mathcal{H}om_H(N,I)\to \mathcal{H}om_H(N_0,I)\to \mathcal{H}om_H(N_1,I)\to 0$$

is an injective resolution of $\mathcal{H}om_H(N,I)$, hence we can calculate $\mathrm{Ext}^r_{G/H}(M,\mathcal{H}om_H(N,I))$ using this resolution.
So since we have a commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_{G/H}(M,\mathcal{H}om_H(N_0,I)) & \xrightarrow{\alpha} & \mathrm{Hom}_{G/H}(M,\mathcal{H}om_H(N_1,I)) \\ & & \\ \mathrm{Hom}_G(\_\otimes_{\mathbb{Z}} N_0,I) & \xrightarrow{\quad\beta\quad} & \mathrm{Hom}_G(\_\otimes_{\mathbb{Z}} N_1,I)\end{array}$$

We conclude that, since $CoKer(\alpha)=\mathrm{Ext}^1_{G/H}(M,\mathcal{H}om_H(N_0,I))$ and, since $Hom_G(\_,I)$ is exact, $CoKer(\beta)=\mathrm{Hom}_G(\mathrm{Tor}_1^{\mathbb{Z}}(M,N),I)$, the canonical iso is induced by the diagram.

Theorem C.8.6. *Let H be a closed normal subgroup of G, and let N and P be G-modules. Then, for any G/H-module M, we have*

$$RHom_{G/H}(M,R\mathcal{H}om_H(N,P))\cong RHom_G(M\otimes^L_{\mathbb{Z}} N,P)$$

Proof. Since by lemma C.8.4 we have

$$\mathrm{Hom}_G(M\otimes_{\mathbb{Z}} N,P)=\mathrm{Hom}_{G/H}(M,\mathcal{H}om_H(N,P))$$

And by lemma C.8.5 we have that $\mathcal{H}om_H(\_,\_)$ maps injectives and flats into acyclics.

Appendix D

An application: Rationality of L-functons

D.1 Frobenius

From now on p is a prime, $q = p^f$ for some f, ℓ is a prime different from p, $\mathbb{F}_q$ is the finite field of order q and $\mathbb{F}$ is its algebraic closure.
X_0 an object defined over $\mathbb{F}_q$ and X its extension to $\mathbb{F}$ (e.g., if $\mathfrak{F}_0$ is a sheaf on a scheme X_0 on $\mathbb{F}_q$, then $\mathfrak{F}$ is the extension of $\mathfrak{F}_0$ on $X = X_0 \times_{\mathbb{F}_q} Spec(\mathbb{F})$).
We denote by Fr_0 the Frobenius endomorphism on X_0, i.e. the identity on the topological space, and locally on the sheaf $Fr_x(t) = t^q$, and by Fr its extension. On $X(\mathbb{F}) = X_0(\mathbb{F})$, it acts like the Frobenius endomorphism of $Gal(\mathbb{F}/\mathbb{F}_q)$.

Frobenius and base change Consider $U \xrightarrow{\pi} X$ an X scheme, then we have a natural map $Fr_{U|X} : U \to U \times_X X$ (here, X is seen as an X-scheme via Fr_X)

Lemma D.1.1. *If π is unramified, then $Fr_{U|X}$ is unramified and injective. If π is Ãľtale, $Fr_{U|X}$ is an isomorphism*

Proof. Let U/X unramified. Then $pr_2 : U \times_X X \to X$ is unramified, and since $pr_2 Fr_{U|X} = \pi$, $Fr_{U|X}$ has discrete fibers and since if $K \subseteq L_0 \subseteq L$ is a tower of field with L/K and L/L_0 unramified, then L/L_0 is unramified, hence $Fr_{U|X}$ is unramified, and since Fr_U is the identity on the topological space, $Fr_{U|X}$ is injective.
If now π is Ãľtale, consider $x \in X$ and $z \in pr_2^{-1}(x)$, hence $k(z) = L$ is finite separable over $k(x) = K$. Take now $y \in \pi^{-1}(x)$, then $\mathcal{O}_{U,y}$ is a flat $\mathcal{O}_{X,x}$-module, hence

$$0 \to \mathcal{O}_{U,y} \otimes_{\mathcal{O}_{X,x}} K \to \mathcal{O}_{U,y} \otimes_{\mathcal{O}_{X,x}} L$$

is exact and finite, hence we have an induced surjective map

$$\begin{tikzcd} Spec(\mathcal{O}_{U,y} \otimes_{\mathcal{O}_{X,x}} \ell) \arrow[r, two heads] \arrow[d, hook] & Spec(\mathcal{O}_{U,y} \otimes_{\mathcal{O}_{X,x}} \kappa) \arrow[d, equal] \\ Fr_{U|X}^{-1}(\{z\}) & \{y\} \end{tikzcd}$$

hence $Fr_{U|X}$ is surjective, hence an isomorphism.

Frobenius correspondence If $\mathfrak{F}_0$ is an abelian sheaf on X_0, then by previous lemma $Fr_0^*\mathfrak{F}_0 \cong \mathfrak{F}_0$, so we have an endomorphism (the Frobenius correspondence):

$$Fr^* : \mathfrak{F} \to \mathfrak{F}$$

which extends to an endomorphism (denoted again by Fr^*) on $H_c^i(X, \mathfrak{F})$.

Remark D.1.2. Frobenius correspondence is functorial in X_0 and $\mathfrak{F}_0$, in the sense that if $X_0 \xrightarrow{f_0} Y_0$ is a morphism and $u \in Hom(f_0^*\mathfrak{F}, \mathfrak{G}) = Hom(\mathfrak{F}, f_{0,*}\mathfrak{G})$, then the following diagrams commute:

$$\begin{tikzcd} X_0 \arrow[r, "Fr_X"] \arrow[d, "f_0"] & X_0 \arrow[d, "f_0"] \\ Y_0 \arrow[r, "Fr_Y"] & Y_0 \end{tikzcd} \qquad \begin{tikzcd} f_0^*\mathfrak{F}_0 \arrow[r, "f_0^*Fr_X^*"] \arrow[d, "u"] & f_0^*\mathfrak{F}_0 \arrow[d, "u"] \\ \mathfrak{G}_0 \arrow[r, "Fr_Y^*"] & \mathfrak{G}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{F}_0 \arrow[r, "Fr_X^*"] \arrow[d, "u"] & \mathfrak{F}_0 \arrow[d, "u"] \\ f_{0*}\mathfrak{G}_0 \arrow[r, "f_{0*}Fr_Y^*"] & f_{0*}\mathfrak{G}_0 \end{tikzcd}$$

(see [Del]) for details

D.2 Trace functions

D.3 Noncommutative Rings

Let Λ be a unitary not necessarily commutative ring, let $\Lambda^\natural$ be the quotient of the additive group of Λ by the subgroup generated by $(ab - ba)$. If $f = (f_i) : \Lambda^n \to \Lambda^n$ is a morphism of free left Λ-mod of finite type, we can denote by $Tr(f)$ the image of $\sum_i f_i$ in $\Lambda^\natural$.

Remark D.3.1. If $\Lambda^n \xrightarrow{f} \Lambda^m$ and $\Lambda^m \xrightarrow{g} \Lambda^n$, we have $Tr(fg) = Tr(gf)$ following trivially from the commutative case.

If now f is an endomorphism of projective Λ-mod of finite type P, then we can choose P' and an iso $\alpha : P \oplus P' \cong \Lambda^n$, hence a section $a : P \to \Lambda^n$ and a retraction $b : \Lambda^n \to P$. Consider $f' = \alpha(f \oplus 0)\alpha^{-1} = afb$, it is an endomoprhism of Λ^n, hence we can define $Tr(f) := Tr(f')$. It does not depend on a, b, in fact if c, d are different morphism, since $dc = id$ and $ba = id$ one has $a = adcba$ and we already know that on free modules $Tr(fg) = Tr(gf)$, so

$$Tr(afb) = Tr(adcbafb) = Tr(cbafbad) = Tr(cfd)$$

If now f is an endomorphism of a projective $\mathbb{Z}/2\mathbb{Z}$-graded Λ-modules (i.e. $P = P_0 \oplus P_1$ with $\Lambda P_i \subseteq P_{i+1}$ with indexes in $\mathbb{Z}/2\mathbb{Z}$), we have components $f_i^j : P_j \to P_i$, we have

$$Tr(f) = Tr(f_0^0) - Tr(f_1^1)$$

If now f is an endomorphism of a projective $\mathbb{Z}/2\mathbb{Z}$-graded Λ-modules filtered with a finite filtration compatible with the graded structure (i.e. $P = P^{(0)} \oplus P^{(1)}$ with $\Lambda P^{(i)} \subseteq P^{(i+1)}$ with indexes in $\mathbb{Z}/2\mathbb{Z}$ and $P^{(j)} = P_1^{(j)} \supseteq ...P_k^{(j)}$ with $\Lambda P_i^{(j)} \subseteq P_i^{(j+1)}$), then

$$Tr(f, P) = \sum Tr(f, Gr_i(P))$$

In general, if f is a morphism of complexes of projective Λ-modules, then

$$Tr(f) = \sum (-1)^i (Tr(f^i, K^i))$$

and in part. if f is null-homotopic, then $Tr(f) = Tr(dH + Hd) = 0$

D.3.1 On the derived category

Let $K^b_{parf}(\Lambda)$ be the full subcategory of $K^b(\Lambda)$ with objects complexes of projective Λ-modules of finite type. The inclusion $K_{parf}(\Lambda) \to D(\Lambda)$ is fully faithful and we can denote $D^b_{parf}(\Lambda)$ the essential image. So we can define a trace on $D^b_{parf}(\Lambda)$ using the definition above, since it does not depend on the homotopy class.
We can do the same thing on $KF_{parf}(\Lambda)$ of the filtered complexes, and get that $DF_{parf}(\Lambda)$ is the category of filtered complexes such that for all p $Gr^p_F(K) \in D^b_{parf}(\Lambda)$. Then

$$Tr(f, K) = \sum_p (Tr_f, Gr^p_F(K))$$

We can also do the same thing for sheaves: if X is a scheme, Λ a ring, then $D^b_{ctf}(X, \Lambda)$ is the full subcategory of $D^-(X, \Lambda)$ whose objects are quasi-isomorphic to bounded complexes of constructible flat sheaves of Λ-modules.
Recall that a complex $K \in D^-(\Lambda)$ is said to have Tor-dimension $\leq r$ if $\forall\ i < -r$ and $\forall\ N$ right Λ-modules we have

$$\mathbb{T}\mathrm{or}_i(N, K) = H^i(N \otimes^{\mathbb{L}}_{\Lambda} K) = 0$$

For the complexes of sheaves of Λ-modules in $D^-(X, \Lambda)$[1], we consider the tor dimension with respect to *constant* sheaves of Λ-modules.

Lemma D.3.2. *Let Λ be a left-Noetherian ring. If $K^\bullet$ a complex of Λ-modules (resp. sheaves of Λ-modules) such that $H^i(K^\bullet)$ are of finite type (resp constructible) and zero for $i >> 0$, then there exists a quasi-isomorphism $K' \xrightarrow{\sim} K$ with K' bounded above with component free of finite type (resp. constructible) and flat.*

[1] The problem here is that $Sh(X, \Lambda)$ has not enough projectives, but one can show that there are enough flat objects and the derived functor does not depend on the flat resolution. See [Har]

Proof. For m such that $H^i(K) = 0$ for $i \geq m$, we can consider $K'^i = 0$ for $i \geq m$. So we need to use induction: we are in the situation of having K'^i $\forall$ $i > n$:

$$\begin{tikzcd} K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots \\ & & K'^{n+1} \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

Such that $H^i(K) \xrightarrow{\sim} H^i(K')$ for $i > n+1$ and $Ker(\tilde{d}) \twoheadrightarrow H^{n+1}(K)$. Hence we can construct by pullbacks:

$$\begin{tikzcd} K^n \arrow[r] & K^n/Im(d) \arrow[r] & Ker(d) \arrow[r] & H^{n+1}(d) \arrow[r] & 0 \\ A \arrow[u] \arrow[r, "u"] & B \arrow[u] \arrow[r] & Ker(d') \arrow[u] \arrow[ur, two heads] & & \end{tikzcd}$$

Hence u is an epi (pullback of an epi in abelian category) and the bottom-line sequence is exact. Since Λ is noetherian, $Ker(d')$ is of finite type (resp. constructible), $H^{n+1}(d)$ is of finite type (resp. constructible) by hypothesis and

$$Ker(B \to Ker(d')) = H^n(d)$$

by pullback-rule, also of finite type (resp. constructible) by hypothesis, hence B is of finite type (resp. constructible).
To conclude, one should take $v : K'^n \to A$ such that uv is epi with K'^n satisfying the hypothesis: for Λ-mod, it is enough to take a free augmentation of finite type $\Lambda^j \twoheadrightarrow B$, since a free module is projective one can lift to $\Lambda^j \xrightarrow{v} B$ such that uv is the augmentation map, which is surjective.
In the case of sheaves, one has that if $S \subseteq \{\phi : U \to X \text{ étale}\}$, then we have a diagram

$$\begin{tikzcd} A \arrow[r, two heads, "u"] & B \\ \bigoplus_S \phi_! \Lambda \arrow[u, "v"] \arrow[ur] & \end{tikzcd}$$

and since B is constructible, Λ is Noetherian and X is Noetherian, there is a finite S and finite J_ϕ such that uv is epi.

Lemma D.3.3. *Let X be a Noetherian scheme, Λ a left-Noetherian ring, $K \in D^-(\Lambda)$ (resp $K \in D^-(X, \Lambda)$), then $K \in D^b_{parf}(\Lambda)$ (resp $K \in D^b_{ctf}(X, \Lambda)$) if and only if K has finite Tor-dimension and $H^i(K)$ are of finite type (resp. K has finite Tor-dimension and $H^i(K)$ are constructible).*

Proof. " $\Rightarrow$ " is trivial: since Λ and X are Noetherian $H^i(K)$ is of finite type (resp. constructible) since K is a complex of modules of finite type (resp. constructible sheaves), and they are also flat, so $\mathbb{T}\mathrm{or}_i(N, K) = 0$ for $i \neq 0$, N Λ-mod (resp. $\mathbb{T}\mathrm{or}_i(N, K) = 0$ for $i \neq 0$, N sheaf of Λ-mod)
" $\Leftarrow$ " Since Λ is noetherian, by previous lemma we can take K' a complex of free modules

of finite type (resp. flat and constructible) quasi-isomorphic to K, so we can suppose K^n free of finite type (resp. flat and constructible). We need to show that it is bounded below: if K has Tor-dimension $\leq r$, we have $H^i(K) = 0$ for $i < -r$ (take $N = \Lambda$, resp $N = \oplus\phi_!(\Lambda)$). In particular, we have a flat resolution

$$.. \to K^{-r-1} \to K^{-r} \to K^{-r}/Im(d) \to 0$$

And for all N and $n \geq 1$

$$\mathrm{Tor}_n(N, K^{-r}/Im(d)) = \mathbb{T}\mathrm{or}_n(N, K^{-r}/Im(d)) = H^{-n-r}(N \otimes^{\mathbb{L}}_{\Lambda} K) = 0$$

Hence $K^{-r}/Im(d)$ is flat of finite presentation, hence projective of finite type (resp. flat constructible). So we have

$$0 \to K^{-r}/Im(d) \to K^{-r+1}...$$

is quasi-isomorphic to K, and it is bounded below, hence it is bounded, so it is in $D^b_{parf}(\Lambda)$ (resp. in $D^b_{ctf}(\Lambda)$) □

Theorem D.3.4. *Let $X \xrightarrow{f} Y$ a separated morphism of finite type between Noetherian schemes. If $K \in D^b_{ctf}(X, \Lambda)$, then $Rf_!(K) \in D^b_{ctf}(Y, \Lambda)$*

Proof. Consider a compactification $X \to \overline{X} \xrightarrow{\bar{f}} Y$. Since $\bar{f}$ is a proper morphism, f_* has finite cohomological dimension (consequence of the proper base change and Tsen theorem, [Del, Arcata IV, 6.1]) and so it defines a functor

$$R\bar{f}_* : D^-(\overline{X}, \Lambda) \to D^-(\overline{Y}, \Lambda)$$

By composition with $j_! : D^-(X, \Lambda) \to D^-(\overline{X}, \Lambda)$, we can define $Rf_!$ on the whole $D^-(X, \Lambda)$. So we have the hypercohomology spectral sequence

$$E_2^{pq} = R^p f_! H^q(K) \Rightarrow \mathbb{R}^{p+q}(f_! K) = H^{p+q}(Rf_! K)$$

And since $H^q(K)$ is constructible, since $R^p f_! H^q(K)$ is constructible ([Del, Arcata IV, 6.2]) we conclude that the cohomology of $Rf_! K$ is constructible.
Suppose that the Tor dimension of K is $\leq -r$, take N a constant sheaf, then the spectral sequence

$$R^p f_*(H^q(N \otimes^{\mathbb{L}} K)) \Rightarrow \mathbb{R}^{p+q} f_*(N \otimes^{\mathbb{L}} K) = H^{p+q}(Rf_*(N \otimes^L K))$$

On the second page is zero for $q \leq r$, hence $H^i(Rf_*(N \otimes^L K)) = 0$ for $i > q$ We now can conclude that $Rf_! K$ has finite Tor dimension after this lemma:

Lemma D.3.5. *For all constant sheaves of right Λ modules, one has*

$$N \otimes^{\mathbb{L}} Rf_! K \xrightarrow{\sim} Rf_!(N \otimes K)$$

Proof. a) Since $j_!$ is exact, one only have to prove it for $\bar{f}_*$, then one can suppose f proper.

b) Considering an acyclic complex quasi-isomorphic to K, we have $Rf_* K \sim f_* K$, so we can work with bounded above complexes since f_* is of finite cohomological dimension.

c) If N is free, we have

$$R^p f_*(N \otimes K^q) \cong N \otimes_\Lambda R^p f_* K^q$$

So $N \otimes K^q$ is acyclic for f_* and $f_*(N \otimes K^p) \cong L \otimes f_* K^q$

d) Taking N_* a free resolution of N, we get $N \otimes^L K \sim Tot(N_* \otimes K)$. Hence

$$Rf_*(N \otimes^L K) \sim Rf_* Tot(N_* \otimes K) \sim Tot(N_* \otimes f_* K) \sim N \otimes^L Rf_* K$$

In particular if Y is the spectrum of a sep.closed filed, then

$$Rf_! = R\Gamma_c : D^b_{ctf}(X, \Lambda) \to D^b_{perf}(\Lambda)$$

D.4 $\mathbb{Q}_\ell$-sheaves

A $\mathbb{Z}_\ell$-sheaf $\mathfrak{F}$ is a projective system of sheaves $\{\mathfrak{F}_n\}$ such that $\mathfrak{F}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$-modules such that

$$\mathfrak{F}_n \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to \mathfrak{F}_{n-1}$$

is an isomorphism. $\mathfrak{F}$ is *lisse* if $\mathfrak{F}_n$ are locally constant. It can be shown ([Del]) that any $\mathbb{Z}_\ell$-sheaf on a Noetherian scheme is locally lisse.
The stalk in a geometric point x of $\mathfrak{F}$ is the $\mathbb{Z}_\ell$-moule $\mathfrak{F}_x = \varprojlim \mathfrak{F}_{n,x}$ From now on, I will denote $\mathfrak{F}_n$ as $\mathfrak{F} \otimes \mathbb{Z}/\ell^{n+1}\mathbb{Z}$. This should be intended in the sense of the previous definition.
Using Artin-Rees, one can show that the category of $\mathbb{Z}_\ell$-sheaves is closed by kernels ([Del]), and clearly it is closed by cokernels.

Remark D.4.1. Recall that a sheaf $\mathfrak{F}$ is locally constant constructible if and only if it is represented by a finite Ãľtale covering $V \to X$. If X is connected and $\bar{x}$ is a geometric point, then the stalk in $\bar{x}$ induces an equivalence between the category of the sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules constructible locally constant and the category of $\mathbb{Z}/\ell^n$-modules of finite type with a continuous action of $\Pi_1(X, x)$. This is given by the restriction of the equivalences:

$$\begin{tikzcd}
FEt_X \arrow[r, "(_)\times_X\{x\}"] \arrow[d, "Yoneda"] & \Pi_1(X,x)set^f \\
\{\text{l.c.c sheaves}\} \arrow[ur, "(_)_x"] &
\end{tikzcd}$$

So we have, by passing to limit

Proposition D.4.2. *If X is connected and $\bar{x}$ is a geometric point, then the stalk in $\bar{x}$ induces an equivalence between the category of the $\mathbb{Z}_\ell$-sheaves lisse and the $\mathbb{Z}_\ell$-modules of finite type with a continuous action of $\Pi_1(X, x)$*

A $\mathbb{Z}_\ell$-sheaf is *torsion free* if the map induced by the multiplication by ℓ is injective. It is torsion if that map is zero.
One can consider the abelian category of $\mathbb{Q}_\ell$-sheaves as the quotient of the $\mathbb{Z}_\ell$-sheaves by the torsion $\mathbb{Z}_\ell$-sheaves. In particular, its objects are $\mathbb{Z}_\ell$-sheaves denoted by $\mathfrak{F} \otimes \mathbb{Q}_\ell$ and the arrows are

$$Hom(\mathfrak{F} \otimes \mathbb{Q}_\ell, \mathfrak{G} \otimes \mathbb{Q}_\ell) := Hom(\mathfrak{F}, \mathfrak{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

The stalk in a geometric point x is the $\mathbb{Q}_\ell$-vector space $\mathfrak{F}_x \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and the cohomology and proper-supported cohomology are defined as

$$H^q_{(c)}(X, \mathfrak{F}) := (\varprojlim H^q_{(c)}(X, \mathfrak{F} \otimes \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Proposition D.4.3. *Let X be separated of finite type on an algebraically closed filed k, then for all $\mathfrak{F}$ $\mathbb{Q}_\ell$ constructible sheaves with $\mathfrak{F} = \mathfrak{F}' \otimes \mathbb{Q}_\ell$ we have $H^p_c(X, \mathfrak{F})$ are finite $\mathbb{Q}_\ell$ vector spaces.*

Proof. Consider a collection

$$K_n = R\Gamma_c(X, \mathfrak{F} \otimes \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \in D_{parf}(\mathbb{Z}/\ell^{n+1}\mathbb{Z})$$

We need to adapt **Lemma 4** to this context: if $\Lambda \to \Lambda'$ is a morphism of noetherian torsion rings, $K \in D_{ctf}(X, \Lambda)$, then we have an iso in $D_{parf}(\Lambda')$

$$R\Gamma_c(X, K) \otimes^{\mathbb{L}}_\Lambda \Lambda' = R\Gamma_c(X, K \otimes^{\mathbb{L}}_\Lambda \Lambda')$$

The idea is to reduce to the proper case, replace K by a complex acyclic for Γ and with stalks in any geometric point homotopically equivalent to a complex of flat Λ-modules. This gives us $\Gamma(X, K) \sim R\Gamma(X, K)$ and $K \otimes_\Lambda \Lambda' \sim K \otimes^{\mathbb{L}}_\Lambda \Lambda'$. Hence we get

$$R\Gamma(X, K) \otimes^{\mathbb{L}}_\Lambda \Lambda' \to \Gamma(X, K) \otimes^{\mathbb{L}}_\Lambda \Lambda' \to \Gamma(X, K \otimes_\Lambda \Lambda') \leftarrow R\Gamma(X, K \otimes^{\mathbb{L}}_\Lambda \Lambda')$$

In particular, in our case, we get in $D_{parf}(\mathbb{Z}/\ell^n\mathbb{Z})$

$$K_n \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \cong K_{n-1}$$

So we can replace again K_n by complexes of free modules of finite type and the isomorphisms given above by isomorphisms of complexes.
Take now $K = \varprojlim K_n$, it is a bounded complex of free $\mathbb{Z}_\ell$ modules and $K_n \cong K \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^{n+1}\mathbb{Z}$.
Fix now $i \in \mathbb{Z}$. Since each $H^i(K_n)$ is a finite abelian group, we have that the decreasing sequence

$$H^i(K_n) \to H^i(K_{n-1})$$

eventually stabilizes. Hence we have the Mittag-Leffer conditions and $\varprojlim$ is an exact functor. Hence

$$H^i(K) = \varprojlim H^i(K_n)$$

So $\varprojlim H^i(K_n)$ is a $\mathbb{Z}_\ell$-module of finite type, hence

$$H^i_c(X, \mathfrak{F}) = \varprojlim(H^i(K_n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a finite $\mathbb{Q}_\ell$ vector space.

Theorem D.4.4. *If X_0 is a separated scheme of finite type on $\mathbb{F}_q$, Λ a Noetherian torsion ring killed by an integer prime to q. Let $K_0 \in D^bctf(X_0, \Lambda)$ then we have*

$$\sum_{x \in X^{Fr^n}} Tr(Fr^{n,*}, K_x) = Tr(Fr^{n,*}, R\Gamma_c(X, K))$$

Corollary D.4.5. *For all n, let $\mathfrak{G}_0$ be a $\mathbb{Q}_\ell$ sheaf, then we have*

$$\sum_{x\in X^{Fr^n}} Tr(F^{n*},\mathfrak{G}_x) = \sum_i (-1)^i Tr(Fr^{*n}, H^i_c(X,\mathfrak{G}))$$

Proof. Substituting $\mathbb{F}_q$ with $\mathbb{F}_{q^n}$ and X_0 with $X_0 \times_{\mathbb{F}_q} Spec(\mathbb{F}_{q^n})$, we can reduce to the case $n = 1$.
Let $\mathfrak{F}_0$ a $\mathbb{Z}_\ell$ sheaf torsion free such that $\mathfrak{G} = \mathfrak{F}\otimes\mathbb{Q}_\ell$, and let $K_n = R\Gamma_c(X,\mathfrak{F}\otimes\mathbb{Z}/\ell^{n+1}\mathbb{Z})$. We have the induced endomorphisms:

$$Fr^* : K_n \to K_n$$

which are deduced one by the other via the isomorphisms. We can replace K_n with quasi isomorphic complexes such that Fr^* is in fact an endomorphism of complexes. Again we have

$$H^i_c(X,\mathfrak{G}) = H^i(K)\otimes\mathbb{Q}_\ell = H^i(K\otimes\mathbb{Q}_\ell)$$

Hence seeing K_n and $K^*\otimes\mathbb{Q}_\ell$ as filtered complex with filtration given by cycles and boundaries, we get:

$$\sum_i (-1)^i Tr(Fr^{*n}, H^i_c(X,\mathfrak{G})) = Tr(Fr, K^*\otimes\mathbb{Q}_\ell) = Tr(Fr,K^*) = \varinjlim Tr(Fr,K^*_n) =$$

$$\varinjlim Tr(Fr, R\Gamma(X,\mathfrak{F}\otimes\mathbb{Z}/\ell^{n+1}\mathbb{Z}))$$

We can use **Theorem 2**:

$$Tr(Fr, R\Gamma(X,\mathfrak{F}\otimes\mathbb{Z}/\ell^{n+1}\mathbb{Z})) = \sum_{x\in X^F} Tr(Fr^*,\mathfrak{F}_x\otimes\mathbb{Z}_\ell^{n+1}) = Tr(Fr^*,\mathfrak{F}_x) \mod \ell^{n+1}$$

Passing to the limit we have the result

D.5 L-functions

Let X_0 be a scheme of finite type over $\mathbb{F}_q$, $q = p^f$, $|X_0|$ the set of its closed points, $\mathfrak{F}_0$ a constructible $\mathbb{Q}_\ell$-sheaf.

Definition D.5.1. We have the L-function associated to $\mathfrak{F}_0$ given by

$$L(X_0,\mathfrak{F}_0) = \prod_{x\in|X_0|} det(1 - Fr^*_x t^{[k(x):\mathbb{F}_p]},\mathfrak{F})^{-1} \qquad \in \mathbb{Q}_\ell[[t]]$$

Theorem D.5.2. *If X_0 is separated, then*

$$L(X_0,\mathfrak{F}_0) = \prod_i det(1 - Fr^* t^f, H^i_c(X,\mathfrak{F}))^{(-1)^{i+1}}$$

In particular if $H^i_c(X,\mathfrak{F}) = 0$ for $i >> 0$, then $L(X_0,\mathfrak{F}_0)$ is rational.

Proof. Since both series have constant term 1, then we can compare their logarithmic derivative, and we have a formula: let f an endomorphism of a projective module over a commutative ring, then

$$t\frac{d}{dt}\log(1-ft)=\sum_n Tr(f^n)t^n$$

Hence

$$t\frac{d}{dt}\log(L(X_0,\mathfrak{F}_0))=\sum_{x\in|X_0|}\sum_n[k(x):\mathbb{F}_p]Tr(Fr_x^{*n},\mathfrak{F})T^{n[k(x):\mathbb{F}_p]}$$

and changing the order of summation and developing using the points in the extensions:

$$\sum_n t^n\sum_{x\in X^{Fr^n}}Tr(Fr^{n*},\mathfrak{F}_x)$$

On the other hand

$$t\frac{d}{dt}\log\prod_i det(1-Fr^*t^f,H_c^i(X,\mathfrak{F}))^{(-1)^{i+1}}=\sum_n t^n\sum_i(-1)^iTr(Fr^{*,n},H_c^i(X,\mathfrak{F}))$$

And comparing term by term by $Corollary 1$ we conclude.

Bibliography

[AGV72] Michael Artin, Alexandre Grothendieck, and Jean Louis Verdier. *Théorie des topos et cohomologie étale des schémas. Seminaire de géométrie algébrique du Bois-Marie SGA 4*, 1972.

[Art69] Michael F Artin. Algebraic approximation of structures over complete local rings. *Publications Mathématiques de l'IHÉS*, 36:23–58, 1969.

[AT67] Emil Artin and John Torrence Tate. *Class field theory*. WA Benjamin New York, 1967.

[AT11] Marco Abate and Francesca Tovena. *Geometria differenziale*. Springer Science & Business Media, 2011.

[Bie87] Mel Bienenfeld. An étale cohomology duality theorem for number fields with a real embedding. *Transactions of the American Mathematical Society*, 303(1):71–96, 1987.

[Blo86] Spencer Bloch. Algebraic cycles and higher k-theory. *Advances in Mathematics*, 61(3):267–304, 1986.

[CC09] Gaëtan Chenevier and Laurent Clozel. Corps de nombres peu ramifiés et formes automorphes autoduales. *Journal of the American Mathematical Society*, 22(2):467–519, 2009.

[CE16] Henry Cartan and Samuel Eilenberg. *Homological Algebra*, volume 19 of *Princeton Mathematical Series*. Princeton University Press, 2016.

[CF67] John William Scott Cassels and Albrecht Fröhlich. Algebraic number theory. 1967.

[Del] P Deligne. Cohomologie étale, seminaire de géométrie algébrique du bois-marie sga 4 1/2. 569.

[FM12] Mathias Flach and Baptiste Morin. On the weil-étale topos of regular arithmetic schemes. *Documenta Mathematica*, 17:313–399, 2012.

[Fu11] Lei Fu. *Etale cohomology theory*, volume 13. World Scientific, 2011.

[GD61] A Grothendieck and J Dieudonné. Éléments de géométrie algébrique. *Publ. math. IHES*, 1961.

[Har] Robin Hartshorne. *Residues and Duality*. Lecture Notes in Mathematics.

[Lan72] Serge Lang. *Introduction to algebraic geometry*. Addison-Wesley Pub. Co, 1972.

[Maz73] Barry Mazur. Notes on étale cohomology of number fields. *Annales scientifiques de l'École Normale Supérieure*, 6(4):521–552, 1973.

[Mil76] JS Milne. Duality in the flat cohomology of a surface. In *Annales scientifiques de l'École Normale Supérieure*, volume 9, pages 171–201, 1976.

[Mil86] James S Milne. Values of zeta functions of varieties over finite fields. *American Journal of Mathematics*, 108(2):297–360, 1986.

[Mil97] James S Milne. *Class field theory*. lecture notes available at http://www.math.lsa.umich.edu/jmilne, 1997.

[Mil06] J.S. Milne. *Arithmetic Duality Theorems*. BookSurge, LLC, 2006. Available at http://www.math.lsa.umich.edu/jmilne.

[Mil16] James S Milne. *Etale Cohomology (PMS-33)*, volume 33. Princeton university press, 2016.

[MO15] David Mumford and Tadao Oda. *Algebraic Geometry: II*. Hindustan Book Agency, 2015.

[Nar13] Wladyslaw Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer Science & Business Media, 2013.

[Neu13] Jürgen Neukirch. *Algebraic number theory*, volume 322. Springer Science & Business Media, 2013.

[Rud87] Walter Rudin. *Real and complex analysis*. Tata McGraw-Hill Education, 1987.

[RZ00] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer, 2000.

[SD01] Henry Peter Francis Swinnerton-Dyer. *A brief guide to algebraic number theory*, volume 50. Cambridge University Press, 2001.

[Ser62] Jean-Pierre Serre. *Corps locaux*, volume 1296 of *Actualités Sci. Indust.* 1962.

[Sta] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu.

[Tam12] Günter Tamme. *Introduction to étale cohomology*. Springer Science & Business Media, 2012.